VANDERBILT UNIVERSITY.

MATH 196.

Summary of Homogeneous Second Order Linear ODEs with constant coefficients.

A second order homogeneous linear differential equation with constant coefficients is of the form

$$Ay'' + By' + Cy = 0, \tag{1}$$

where A, B, and C are constants. We can assume $A \neq 0$, otherwise this would be a first order equation (which we have already learned how to solve). Dividing the equation by A gives

$$y'' + by' + cy = 0, \tag{2}$$

Remark 1. All formulas here provided assume that the differential equation is written as in (2), i.e., with the coefficient of y'' equal to one. If you are giving an equation where the coefficient of y'' is not equal to one, as in (1), you have first to divide the equation by that same coefficient to write it as in (2).

From (2), write the **characteristic equation**:

$$\lambda^2 + b\lambda + c = 0,$$

whose roots are given by

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}$$
$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

There are three possible cases.

<u>CASE 1:</u> λ_1 and λ_2 are **real and distinct**.

In this case, the functions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two linearly independent solutions of the differential equation (2), and the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

<u>CASE 2:</u> λ_1 and λ_2 are **real and equal**.

Write $\lambda_1 = \lambda_2 = \lambda$. In this case, the functions $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$ are two linearly independent solutions of the differential equation (2), and the general solution is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}.$$

<u>CASE 3:</u> λ_1 and λ_2 are **complex imaginary solutions**.

In this case, write $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where α and β are *real numbers*. The functions $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions of the differential equation (2), and the general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$