VANDERBILT UNIVERSITY
MATH 196 — EXAMPLES OF SECTION 7.5

Question 1. Solve $x' = Ax$, where

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

Solutions.

1. A simple computation gives

$$\det \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} = -\lambda(\lambda-5)^2,$$

so $\lambda_1 = 0$ and $\lambda_2 = 5$ are the eigenvalues, with λ_2 of multiplicity two.

To find an eigenvector associated with λ_1, we solve

$$\left[\begin{array}{ccc:c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination we find $u_1 = (-4, -5, 2)$, and $x_1 = e^{0t}u_1 = (-4, -5, 2)$ is a solution to $x' = Ax$.

Next, we move to λ_2, and consider:

$$\left[\begin{array}{ccc:c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination, we find

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, this system has only one free variable, yielding only one linearly independent eigenvector which we can take to be $u_2 = (-2, 0, 1)$. Hence $x_2 = e^{5t}(-2, 0, 1)$ is a second linearly independent solution to $x' = Ax$. To find a third linearly independent solution, we need to find a generalized eigenvector associated with $\lambda_2 = 5$. Compute

$$(A - 5I)^2 = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Now we solve

$$\left[\begin{array}{rrr:r} -4 & 20 & -8 & 0 \\ -5 & 25 & -10 & 0 \\ 2 & -10 & 4 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination gives

$$\left[\begin{array}{rrr:r} -1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has two free variables that yield two linearly independent generalized eigenvectors $u_2 = (-2, 0, 1)$ and $u_3 = (5, 1, 0)$ (notice that we already knew from above that u_2 is a solution since it is an eigenvector). To find a third (linearly independent) solution to $x' = Ax$, compute

$$x_3 = e^{5t}(u_3 + t(A - 5I)u_3) = e^{5t}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + te^{5t}\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e^{5t}\begin{bmatrix} 5-4t \\ 1 \\ 2t \end{bmatrix}.$$

The general solution is given by $x = c_1x_1 + c_2x_2 + c_3x_3$.

URL: `http://www.disconzi.net/Teaching/MAT196-Spring-15/MAT196-Spring-15.html`