# SOLUTIONS TO GRADED PROBLEMS, 1.6, 3.1

MATH 196.3

**Problem 1** (1.6.15)**.** Find a general solution to the ODE

$$x(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + y(3x+y) = 0$$

*Solution.* Many of you successfully solved this problem; however, I was surprised how many substituted for $\frac{y}{x}$ before that expression made any appearance in the problem. This made for some algebraic contortions.

That strategy will work, but I prefer not to make a substitution out of thin air. If we're looking for a substitution, the one that jumps out to me is $u = x+y$. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$ and our ODE becomes

$$\begin{aligned} xu\left(\frac{\mathrm{d}u}{\mathrm{d}x} - 1\right) + (u-x)(u+2x) = 0 \\ xu\frac{\mathrm{d}u}{\mathrm{d}x} = 2x^2 - u^2 \end{aligned}$$

Now this looks like a homogeneous equation!

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\frac{x}{u} - \frac{u}{x}$$

With $v = \frac{u}{x}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$ we get

$$\begin{aligned} x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{2}{v} - 2v = \frac{2-2v^2}{v} \\ \frac{2v}{v^2-1}\mathrm{d}v &= -\frac{4}{x}\mathrm{d}x \\ \log(v^2-1) &= -4\log(x) + C \\ v^2 - 1 &= \frac{C}{x^4} \end{aligned}$$

Now substituting back in for $v$, then for $u$, we get

$$\begin{aligned} \frac{(x+y)^2}{x^2} - 1 &= \frac{C}{x^4} \\ x^2\left((x+y)^2 - x^2\right) &= C \end{aligned}$$

□

**Problem 2.** Show that the ODE

$$\left(x^3 + \frac{y}{x}\right)\mathrm{d}x + \left(y^2 + \log(x)\right)\mathrm{d}y = 0$$

is exact, and give its general solution.

*Proof.* We use the theorem given in the text to test for exactness: since

$$\partial_y \left( x^3 + \frac{y}{x} \right) = \frac{1}{x} = \partial_x \left( y^2 + \log(x) \right)$$

the equation is exact.

It is *only* once we have verified this that we can conclude that there *exists* a function $F(x, y)$ whose partial derivatives match the terms in the original ODE. Once we know that $F$ exists, of course, we actually find it by integrating:

$$\int x^3 + \frac{y}{x} \, \mathrm{d}x = \frac{1}{4}x^4 + y \log(x) + C_1(y)$$

$$\int y^2 + \log(x) \, \mathrm{d}y = \frac{1}{3}y^3 + y \log(x) + C_2(x)$$

so we can write the general solution as

$$F(x, y) := \frac{1}{4}x^4 + y \log(x) + \frac{1}{3}y^3 = C$$

where $C$ is now a true constant (i.e. with respect to both $x$ and $y$). □

**Problem 3** (3.1.11)**.** Your solutions to this problem were overall excellent; I only want to remind you that it is much easier to read (and check) row reductions when you include indications to the reader of what operations you made use of at each step. Prof Disconzi suggested one possible notation for these indications in class; you're free to use others, but whatever you do should point your reader in the right direction to follow your footsteps.