VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
EXAMPLES OF SECTIONS 4.1 AND 4.2.

Question 1. Determine whether the vectors $(5, -2, 4)$, $(2, -3, 5)$, and $(4, 5 - 7)$ are linearly independent or dependent.

Question 2. Consider the set V of all triples (x, y, z) such that $x = 3$. Is V a vector space?

Question 3. Find solution vectors $\vec{u}$ and $\vec{v}$ such that the solution space is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$:

$$\begin{cases} x_1 & - & 4x_2 & - & 3x_3 & - & 7x_4 & = & 0 \\ 2x_1 & - & x_2 & + & x_3 & + & 7x_4 & = & 0 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 11x_4 & = & 0 \end{cases}$$

SOLUTIONS.

1. Denote the vectors by $\vec{u} = (5, -2, 4)$, $\vec{v} = (2, -3, 5)$, and $\vec{w} = (4, 5 - 7)$. Consider

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}.$$

Recall that the vectors are linearly independent if the only solution of the previous equation is $a = b = c = 0$, and linearly dependent otherwise. The equation can be written as

$$a \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + c \begin{bmatrix} 4 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or in matrix form

$$\begin{bmatrix} 5 & 2 & 4 \\ -2 & -3 & 5 \\ 4 & 5 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The system will have a unique solution provided that the matrix of the system is invertible. But we readily check that

$$\det \begin{bmatrix} 5 & 2 & 4 \\ -2 & -3 & 5 \\ 4 & 5 & -7 \end{bmatrix} = 0,$$

which means that the matrix is not invertible, hence the system does not have a unique solution, and therefore the vectors are linearly dependent.

2. First notice that elements of V can be written as $(3, y, z)$. In order for V to be a vector space, there must exist a zero element, i.e., an element $q = (q_1, q_2, q_3)$ such that $q \in V$ and $q + u = u$ for every $u \in V$. But if $q \in V$ then it can be written as $q = (3, q_2, q_3)$, and it follows that

$$q + u = (3, q_2, q_3) + (3, u_2, u_3) = (6, q_2 + u_2, q_3 + u_3) \neq (3, u_2, u_3).$$

Therefore V it is not a vector space.

3. The augmented matrix of the system is

$$\begin{bmatrix} 1 & -4 & -3 & -7 & \vdots & 0 \\ 2 & -1 & 1 & 7 & \vdots & 0 \\ 1 & 2 & 3 & 11 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination we find

$$\begin{bmatrix} 1 & 0 & 1 & 5 & \vdots & 0 \\ 0 & 1 & 1 & 3 & \vdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore x_3 and x_4 are free variables. Denoting by $x_3 = s$, $x_4 = t$, we can then write

$$x_1 = -s - 5t,$$
$$x_2 = -s - 3t.$$

Therefore solutions $\vec{x} = (x_1, x_2, x_3, x_4)$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - 5t \\ -s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} = s\vec{u} + t\vec{v},$$

where

$$\vec{u} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$