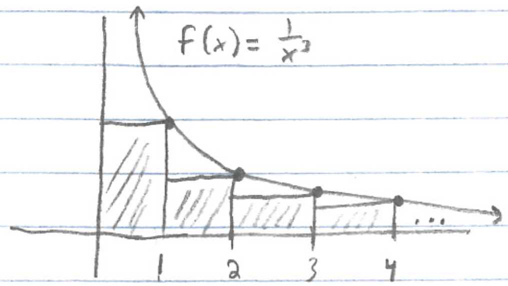
10/24

11.3

Example: $\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots$



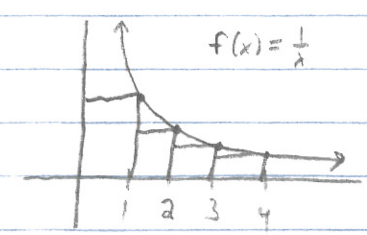
$$\sum_{n=2}^{\infty} \frac{1}{n^3} \le \int_1^{\infty} \frac{1}{x^3}\,dx \quad \text{(if the integral converges)}$$

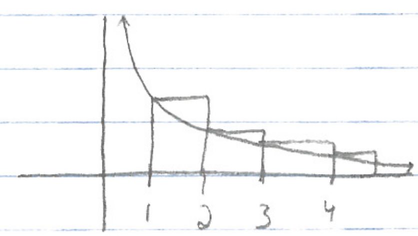
$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left. \frac{-1}{2x^2} \right|_1^t = \lim_{t\to\infty} \left(\frac{-1}{2t^2} + \frac{1}{2} \right) = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^3} \le 1 + \frac{1}{2} = \frac{3}{2}$$

The series converges to some value less than $\frac{3}{2}$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$





$\int_1^{n+1} \frac{1}{x}\,dx \le \sum_{i=1}^{n} \frac{1}{i}$; since $\int_1^{\infty} \frac{1}{x}\,dx$ diverges, so does the series

Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then,

i.) If $\int_1^{\infty} f(x)\,dx$ converges, so does $\sum_{n=1}^{\infty} a_n$.

ii.) If $\int_1^{\infty} f(x)\,dx$ diverges, so does $\sum_{n=1}^{\infty} a_n$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$

$$\int_1^{\infty} \frac{1}{x^2+6x+13}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(x+3)^2+4}\,dx = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)\Big|_1^t =$$

$$\lim_{t\to\infty} \left(\frac{1}{2}\tan^{-1}\left(\frac{t+3}{2}\right) - \frac{1}{2}\tan^{-1}(2)\right) = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(2)\right)$$

By the Integral Test, the series converges (to some value less than $\frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(2)\right) + \frac{1}{20}$.)

p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

- $p<0$: $\lim_{n\to\infty} \frac{1}{n^p} = \infty$, so the series diverges
- $p=0$: $\lim_{n\to\infty} \frac{1}{n^p} = 1$, so the series diverges
- $p>1$: $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^t =$

  $$\lim_{t\to\infty} \left(\frac{1}{1-p} t^{1-p} - \frac{1}{1-p}\right) = \frac{1}{p-1}$$

  By the Integral Test, the series converges (to some value less than $1 + \frac{1}{p-1} = \frac{p}{p-1}$.)

- $0<p\le 1$: $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges (p-test), so the series diverges (Integral Test)

$$\left[\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < 2\right]$$

Example: $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

$$\int_1^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \frac{1}{3}\int_2^{t^3+1} \frac{du}{u} =$$

$$\frac{1}{3}\lim_{t\to\infty} \ln|u|\Big|_2^{t^3+1} = \frac{1}{3}\lim_{t\to\infty} \left(\ln|t^3+1| - \ln 2\right) = \infty$$

Let $u = x^3+1$.
Then, $du = 3x^2\,dx$.

By the Integral Test, the series diverges.

11.4

Example: $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2+6n+13}$

For each n, $0 \le \frac{\sin^2(n)}{n^2+6n+13} \le \frac{1}{n^2+6n+13}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$ converges (by the Integral Test), $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2+6n+13}$ converges as well (by the Comparison Test).