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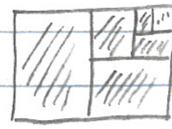
11.2

Recall $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} \frac{1}{2^i}}_{} = 1.$



nth partial sum $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$

Example: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots \quad , \quad -1 < r < 1$ [geometric series]

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} = a(1 + r + r^2 + \ldots + r^{n-1})$$
$$rs_n = a(r + r^2 + r^3 + \ldots + r^n)$$

$$(1-r)s_n = a(1-r^n) \quad , \quad s_n = a\left(\frac{1-r^n}{1-r}\right)$$

$\lim_{n \to \infty} s_n = \lim_{n \to \infty} a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r}$, series converges to $\frac{a}{1-r}$

Example: $3 - \frac{6}{5} + \frac{12}{25} - \frac{24}{125} + \ldots \quad ; \quad a = 3, r = \frac{-2}{5}$

Since $|r| < 1$, the series converges to $\frac{3}{1-(\frac{-2}{5})} = \frac{3}{(\frac{7}{5})} = \frac{15}{7}$.

Example: $\sum_{n=0}^{\infty} \frac{e^{2n}}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{e^2}{5}\right)^n$

Since $\frac{e^2}{5} > 1$, the series diverges.

Example: For $|x| < 1$, $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \ldots$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \; ; \; 1 = A(n+2) + Bn = (A+B)n + 2A \; ; \; A = \frac{1}{2}, B = \frac{-1}{2}$$

$$s_n = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \frac{1}{2}\left(\frac{1}{n-2} - \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$s_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \; ; \; \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{4}.$$

Example: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ [harmonic series - Examples]

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$.

Proof: $\lim_{n\to\infty} s_n = s$, where $s_n = \sum_{i=1}^{n} a_i$.

$a_n = s_n - s_{n-1}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0$

Example: $\sum_{n=1}^{\infty} \frac{n^3}{4n^3 - 8}$

Since $\lim_{n\to\infty} \frac{n^3}{4n^3-8} = \frac{1}{4}$, the series diverges [Test for Divergence]

Theorem: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge:

1.) $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

2.) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

Example: $\sum_{n=2}^{\infty} \left(\frac{4}{n(n+2)} + \frac{1}{3^n}\right)$

$\sum_{n=2}^{\infty} \frac{4}{n(n+2)} = \sum_{n=1}^{\infty} \frac{4}{n(n+2)} - \frac{4}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n(n+2)} - \frac{4}{3} = \frac{5}{3}$

$\sum_{n=2}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} - \left(1 + \frac{1}{3}\right) = \frac{1}{1-\frac{1}{3}} - \left(1 + \frac{1}{3}\right) = \frac{1}{6}$

So, $\sum_{n=2}^{\infty} \left(\frac{4}{n(n+2)} + \frac{1}{3^n}\right) = \frac{5}{3} + \frac{1}{6} = \frac{11}{6}$