MAT 155B - FALL 12 — SECTIONS 04 AND 13
TEST 1 (100 POINTS TOTAL)

NAME: SOLUTIONS
Mark your section: SECTION 04 (11:10am) SECTION 13 (10:10am)

Question	Points
1 (10 pts)	
2 (15 pts)	
3 (20 pts)	
4 (20 pts)	
5 (20 pts)	
6 (10 pts)	
7 (5 pts)	
Extra Credit (5 pts)	
TOTAL (100 pts)	

Question 1 (10 pts). Find a formula for the inverse of the given functions:

(a) (5 pts) $h(x) = x + \sqrt{x}$.

$y = h(x)$

$y = x + \sqrt{x}$

$(y - x) = \sqrt{x}$

$(y-x)^2 = (\sqrt{x})^2$

$y^2 + x^2 - 2yx = x$

$x^2 - (1+2y)x + y^2 = 0$

$x = \frac{(1+2y) \pm \sqrt{(1+2y)^2 - 4y^2}}{2}$

$x = \frac{(1+2y) \pm \sqrt{1+4y}}{2}$

$x \geq 0 \Rightarrow$ keep $+$ solution:

$x = \frac{(1+2y) + \sqrt{1+4y}}{2}$

$$\boxed{h^{-1}(x) = \frac{1+2x+\sqrt{1+4x}}{2}}$$

(b) (5 pts) $y = \frac{\sqrt[3]{e^{t-2}+5}}{4}$.

$4y = \sqrt[3]{e^{t-2}+5}$

$(4y)^3 - 5 = e^{t-2}$

$$\boxed{t = \ln\left[(4y)^3 - 5\right] + 2}$$

Question 2 (15 pts). Find the derivative of the given functions:

(a) (5 pts) $y = (\ln \arcsin x)^2$.

$$y' = \frac{2 \ln \arcsin x}{\arcsin x} \cdot (\arcsin x)'$$

$$\boxed{y' = \frac{2 \ln \arcsin x}{\arcsin x \sqrt{1 - x^2}}}$$

(b) (10 pts) $g(x) = (\ln \cos x)^{\sin x}$.

$$y = (\ln \cos x)^{\sin x}$$

$$\ln y = \sin x \ln(\ln \cos x)$$

$$\frac{1}{y} y' = \cos x \ln(\ln \cos x) + \frac{\sin x \cdot (-\sin x)}{\ln(\cos x) \cdot \cos x}$$

$$\boxed{y' = (\ln \cos x)^{\sin x}\left(\cos x \ln \cos x - \frac{\sin^2 x}{\cos x \ln(\cos x)}\right)}$$

Question 3 (20 pts). Compute the following indefinite integrals:

(a) (10 pts) $\int \frac{\cos x}{1+\sin^2 x}\,dx$.

$u = \sin x \rightarrow du = \cos x\,dx$

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \boxed{\tan^{-1}(\sin x) + C}$$

(b) (10 pts) $\int \frac{1}{x \ln x}\,dx$.

$u = \ln x \rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{1}{x \ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\ln x| + C}$$

Question 4 (20 pts). The half-life of cesium-137 is 30 years.

(a) (10 pts) Given a sample of cesium-137, find a formula for the amount remaining after t years.

$$y = y_0 e^{kt}$$

$$y(30) = \frac{1}{2}\cancel{y_0} = \cancel{y_0} e^{k\cdot 30} \Rightarrow \frac{1}{2} = e^{k30} \Rightarrow k = \frac{1}{30}\ln(1/2)$$

$$\Rightarrow y = y_0 e^{\frac{t}{30}\ln(1/2)} = y_0 e^{\ln(\frac{1}{2})^{t/30}} = \boxed{y_0\left(\frac{1}{2}\right)^{t/30}}$$

(b) (10 pts) How long does it take for a sample of cesium-137 to decay to 5 % of its initial value?

$$y(t) = 0.05\cancel{y_0} = \cancel{y_0}\left(\frac{1}{2}\right)^{t/30}$$

$$\Rightarrow \ln(0.05) = \frac{t}{30}\ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \boxed{t = \frac{30\ln(0.05)}{\ln(1/2)}}$$

Question 5 (20 pts). The rate of change of atmospheric pressure p with respect to the altitude is proportional to p. At sea level the pressure is 101.3 kPa, and at 1000 m the pressure is equal to 87.14 kPa. What is the pressure at an altitude of 3000 m?

Let h = altitude

$$\frac{dp}{dh} = kp \Rightarrow p = \underbrace{p_0}_{= 101.3 \text{ (measured in kPa)}} e^{kh}$$

$$p = 101.3\, e^{kh}$$

$$p(1000) = 87.14 \text{ kPa} = 101.3\, e^{k \cdot 1000}$$

$$\Rightarrow k = \frac{1}{1000} \ln\left(\frac{87.14}{101.3}\right)$$

$$p = 101.3\, e^{\frac{h}{1000} \ln\left(\frac{87.14}{101.3}\right)} = 101.3\, e^{\ln\left(\frac{87.14}{101.3}\right)^{h/1000}}$$

$$p = 101.3 \left(\frac{87.14}{101.3}\right)^{\frac{h}{1000}}$$

$$p(3000) = 101.3 \left(\frac{87.14}{101.3}\right)^{\frac{3000}{1000}} = \boxed{\frac{(87.14)^3}{(101.3)^2}}$$

Question 6 (10 pts). Compute the following limits.

(a) (3 pts) $\lim_{x\to\infty} x^2 e^{-x^2}$ $= \infty \cdot 0$

$$x^2 e^{-x^2} = \frac{x^2}{e^{x^2}} \xrightarrow[x\to\infty]{} \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cancel{2x}}{\cancel{2x} e^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = \boxed{0}$$

(b) (3 pts) $\lim_{x\to 0^+} x \ln x$

$$x \ln x \xrightarrow[x\to 0^+]{} 0\cdot(-\infty). \quad x \ln x = \frac{\ln x}{1/x} \xrightarrow[x\to 0^+]{} \frac{-\infty}{\infty} \overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} =$$

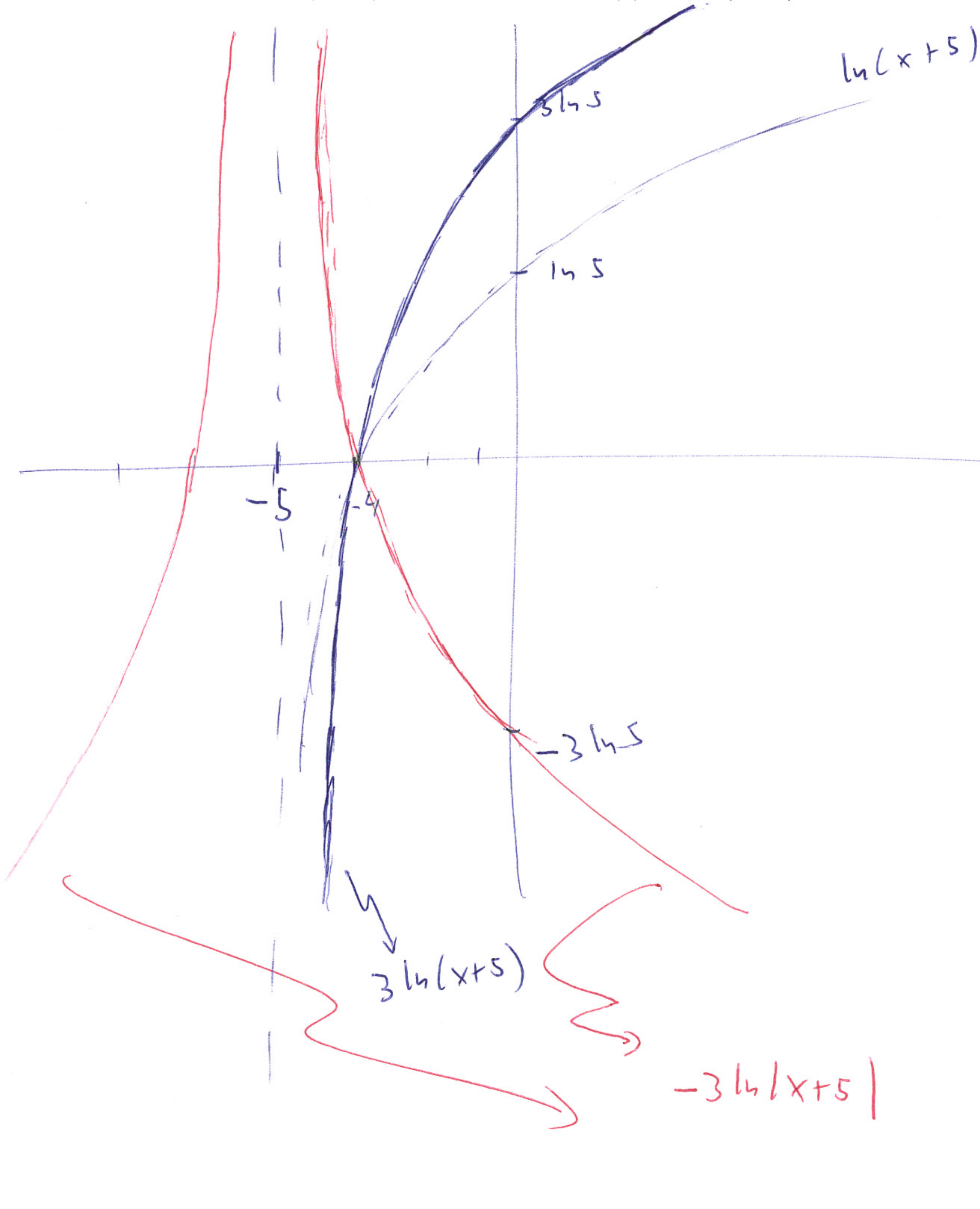
$$= \lim_{x\to 0^+} (-x) = \boxed{0}$$

(c) (4 pts) $\lim_{x\to\infty} x^{\frac{1}{x}}$ $= \infty^0$

$$y = x^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln x \xrightarrow[x\to\infty]{} \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = \boxed{1}$$

Question 7 (5 pts). Graph the function $f(x) = -3\ln|x+5|$.



Extra credit question (5 pts). The *error function* $\mathrm{erf}(x)$ is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Show that

$$\int_a^b e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \left(\mathrm{erf}(b) - \mathrm{erf}(a)\right).$$

From the F.T.C: $\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$

$$\Longrightarrow \int_a^b e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \int_a^b \mathrm{erf}'(x)\, dx$$

$$= \frac{\sqrt{\pi}}{2} \left(\mathrm{erf}(b) - \mathrm{erf}(a) \right)$$