

MAT 155B - FALL 12 – SOLUTIONS TO ASSIGNMENT 3

Question 1. Suppose that a plate is immersed vertically in a fluid with density ρ and the width of the plate is $w(x)$ at a depth of x meters beneath the surface of the fluid. If the top of the plate is at depth a and the bottom is at depth b , show that the hydrostatic force on one side of the plate is

$$F = \int_a^b \rho g x w(x) dx,$$

where g stands for the acceleration of gravity.

Solution. Pick a point $x \in [a, b]$ and consider a horizontal section of the plate through x . Suppose this section defines a thin rectangle of height Δx and width $w(x)$ whose area is $w(x)\Delta x$. The pressure on this thin rectangle is then $\rho g x$, and therefore the hydrostatic force is given by

$$F = PA = \rho g x w(x) \Delta x$$

The total force is then approximated by adding over different horizontal sections:

$$F \approx \sum_{\text{sections}} g x w(x) \Delta x.$$

Taking the limit $\Delta x \rightarrow 0$ yields

$$F = \int_a^b \rho g x w(x) dx.$$

Question 2. In Economics, the *demand function* $p(x)$ is the price that a company has to charge in order to sell x units of a commodity. The graph of $p(x)$ is called a *demand curve*. Denote by x_* the amount of the commodity which is currently available, so that $p_* = p(x_*)$ is the current selling price.

Assume that $p(x)$ is a decreasing function.

(a) What is the interpretation of the quantity

$$\int_0^{x_*} (p(x) - p_*) dx ?$$

Justify your answer.

(b) Is the assumption that $p(x)$ is decreasing reasonable? Justify your answer.

Solution. Consider two points, x and $x + \Delta x$, for some small value Δx . If x units had been available at a price $p(x)$, then consumers who had bought the product when $x + \Delta x$ units were available would have paid $p(x + \Delta x)$, which is less than $p(x)$ since the function p is decreasing. Therefore those buying at a price $p(x + \Delta x)$ save a total of

$$(p(x) - p(x + \Delta x)) \Delta x.$$

Therefore, the integral

$$\int_0^{x_*} (p(x) - p_*) dx$$

represents is the total amount of money saved by consumers in purchasing the commodity at a price p_* .

The assumption of $p(x)$ being decreasing is reasonable since one expects that selling larger quantities requires decreasing the price.