MAT 155B - FALL 12 - SOLUTIONS TO ASSIGNMENT 2

We want to show that

$$\lim_{b\to\infty}\lim_{a\to 0^+}\int_a^b \frac{\arctan(\pi x)-\arctan x}{x}\,dx = \frac{\pi}{2}\ln\pi.$$

We shall provide two different solutions. The first one is longer, but more elementary. The second solution is shorter, but more involved.

Solution 1.
Let us forget about the limits for a moment and compute the definite integral. We can take the limits later. First break the integral in two:

$$\int_a^b \frac{\arctan(\pi x)-\arctan x}{x}\,dx = \int_a^b \frac{\arctan(\pi x)}{x}\,dx - \int_a^b \frac{\arctan x}{x}\,dx. \tag{1}$$

In the first integral, choose $u=\arctan(\pi x)$ and $dv=\frac{1}{x}\,dx$, so

$$u=\arctan(\pi x)\Rightarrow du=\frac{\pi}{1+\pi^2x^2}\,dx,$$
$$dv=\frac{1}{x}\,dx\Rightarrow v=\ln x.$$

It is not necessary to put an absolute value here, i.e, $\ln|x|$, since x is always positive as the domain of integration is $[a,b]$ with $a>0$. Therefore

$$\int_a^b \frac{\arctan(\pi x)}{x}\,dx = \arctan(\pi x)\ln x\Big|_a^b - \pi\int_a^b \frac{\ln x}{1+\pi^2x^2}\,dx. \tag{2}$$

Analogously, in the second integral on the right hand side of (1), choose choose $u=\arctan x$ and $dv=\frac{1}{x}\,dx$, so

$$u=\arctan x\Rightarrow du=\frac{1}{1+x^2}\,dx,$$
$$dv=\frac{1}{x}\,dx\Rightarrow v=\ln x,$$

and then

$$\int_a^b \frac{\arctan x}{x}\,dx = \arctan x\ln x\Big|_a^b - \int_a^b \frac{\ln x}{1+x^2}\,dx. \tag{3}$$

Using (2) and (3) into (1) yields

$$\begin{aligned}\int_a^b \frac{\arctan(\pi x)-\arctan x}{x}\,dx &= \arctan(\pi x)\ln x\Big|_a^b - \pi\int_a^b \frac{\ln x}{1+\pi^2x^2}\,dx\\ &\quad -\left(\arctan x\ln x\Big|_a^b - \int_a^b \frac{\ln x}{1+x^2}\,dx\right)\\ &= \arctan(\pi b)\ln b-\arctan(\pi a)\ln a-\arctan b\ln b+\arctan a\ln a\\ &\quad +\int_a^b \frac{\ln x}{1+x^2}\,dx-\pi\int_a^b \frac{\ln x}{1+\pi^2x^2}\,dx.\end{aligned}$$

Let us now take the limits. First we need to take the limit when $a \to 0^+$:

$$\begin{aligned}
&\lim_{a\to 0^+} \int_a^b \frac{\arctan(\pi x) - \arctan x}{x}\,dx \\
&= \lim_{a\to 0^+} \Big(\arctan(\pi b)\ln b - \arctan(\pi a)\ln a - \arctan b \ln b + \arctan a \ln a \\
&\qquad + \int_a^b \frac{\ln x}{1+x^2}\,dx - \pi \int_a^b \frac{\ln x}{1+\pi^2 x^2}\,dx \Big) \\
&\qquad = \arctan(\pi b)\ln b - \arctan b \ln b + \lim_{a\to 0^+} \Big(-\arctan(\pi a)\ln a + \arctan a \ln a \\
&\qquad + \int_a^b \frac{\ln x}{1+x^2}\,dx - \pi \int_a^b \frac{\ln x}{1+\pi^2 x^2}\,dx \Big).
\end{aligned} \tag{4}$$

Let us compute the first two limits.

$$\lim_{a\to 0^+} (-\arctan(\pi a)\ln a + \arctan a \ln a) = \lim_{a\to 0^+} (-\arctan(\pi a) + \arctan a)\ln a.$$

Multiplying and dividing by a,

$$\lim_{a\to 0^+} (-\arctan(\pi a)\ln a + \arctan a \ln a) = \lim_{a\to 0^+} \Big(\frac{-\arctan(\pi a) + \arctan a}{a} a \ln a \Big). \tag{5}$$

Using L'Hospital rule, we can compute

$$\begin{aligned}
&\lim_{a\to 0^+} \frac{-\arctan(\pi a) + \arctan a}{a} = \frac{0}{0} \\
&\overset{L'H}{=} \lim_{a\to 0^+} \frac{(-\arctan(\pi a) + \arctan a)'}{a'} \\
&= \lim_{a\to 0^+} \Big(-\frac{\pi}{1+\pi^2 a^2} + \frac{1}{1+a^2} \Big) = 1 - \pi
\end{aligned} \tag{6}$$

Also using L'Hospital we find

$$\lim_{a\to 0^+} a \ln a = 0. \tag{7}$$

Using (6) and (7) into (5) gives

$$\lim_{a\to 0^+} (-\arctan(\pi a)\ln a + \arctan a \ln a) = (1-\pi)\cdot 0 = 0,$$

and therefore (4) becomes

$$\begin{aligned}
&\lim_{a\to 0^+} \int_a^b \frac{\arctan(\pi x) - \arctan x}{x}\,dx \\
&= \arctan(\pi b)\ln b - \arctan b \ln b + \lim_{a\to 0^+} \Big(\int_a^b \frac{\ln x}{1+x^2}\,dx - \pi \int_a^b \frac{\ln x}{1+\pi^2 x^2}\,dx \Big).
\end{aligned} \tag{8}$$

But since

$$\lim_{a\to 0^+} \int_a^b f(x)\,dx = \int_0^b f(x)\,dx,$$

provided that the integral exists, we can write (8) as

$$\lim_{a\to 0^+}\int_a^b \frac{\arctan(\pi x)-\arctan x}{x}\,dx = \arctan(\pi b)\ln b - \arctan b \ln b + \int_0^b \frac{\ln x}{1+x^2}\,dx - \pi\int_0^b \frac{\ln x}{1+\pi^2x^2}\,dx. \tag{9}$$

Remark 1. As usual, when you are asked to simply compute some limit/integral etc, it is implicitly assumed that such object exists. Hence here you do not have to show that the limit

$$\lim_{a\to 0^+}\Big(\int_a^b \frac{\ln x}{1+x^2}\,dx - \pi\int_a^b \frac{\ln x}{1+\pi^2x^2}\,dx\Big) \tag{10}$$

exits. But now that we have covered improper integrals in class, you should be able to prove that the above integrals converge.

We want now to take the limit when $b\to\infty$, so that (9) reads

$$\lim_{b\to\infty}\lim_{a\to 0^+}\int_a^b \frac{\arctan(\pi x)-\arctan x}{x}\,dx = \lim_{b\to\infty}\Big(\arctan(\pi b)\ln b - \arctan b \ln b + \int_0^b \frac{\ln x}{1+x^2}\,dx - \pi\int_0^b \frac{\ln x}{1+\pi^2x^2}\,dx\Big). \tag{11}$$

Let us first compute

$$\lim_{b\to\infty}\Big(\arctan(\pi b)\ln b - \arctan b\ln b\Big) = \lim_{b\to\infty}\Big(\arctan(\pi b) - \arctan b\Big)\ln b \tag{12}$$

This is computed similarly to the previous limit $a\to 0^+$. Multiplying and dividing by b,

$$\lim_{b\to\infty}\Big(\arctan(\pi b)\ln b - \arctan b\ln b\Big) = \lim_{b\to\infty}\Big[\Big(\arctan(\pi b) - \arctan b\Big)b\Big]\frac{\ln b}{b}.$$

Since $\arctan x\to\frac{\pi}{2}$ when $x\to\infty$,

$$\begin{aligned}
&\lim_{b\to\infty}\Big(\arctan(\pi b)-\arctan b\Big)b = 0\cdot\infty,\ \text{rewrite as}\\
&\lim_{b\to\infty}\frac{\arctan(\pi b)-\arctan b}{1/b} = \frac{0}{0} \overset{L'H}{=} \lim_{b\to\infty}\frac{(\arctan(\pi b)-\arctan b)'}{(1/b)'}\\
&= \lim_{b\to\infty}\frac{\frac{\pi}{1+\pi^2b^2}-\frac{1}{1+b^2}}{-\frac{1}{b^2}} = \lim_{b\to\infty}\Big(-\frac{b^2}{1+\pi^2b^2}+\frac{b^2}{1+b^2}\Big) = \frac{\pi^2-1}{\pi^2}.
\end{aligned}\tag{13}$$

Using the L'Hospital rule once more we also find

$$\lim_{b\to\infty}\frac{\ln b}{b} = 0, \tag{14}$$

and therefore, combining (13), (14) and (12) gives

$$\lim_{b\to\infty}\Big(\arctan(\pi b)\ln b - \arctan b\ln b\Big) = \frac{\pi^2-1}{\pi^2}\cdot 0 = 0.$$

We conclude therefore that (11) becomes

$$\lim_{b\to\infty}\lim_{a\to 0^+}\int_a^b \frac{\arctan(\pi x)-\arctan x}{x}\,dx = \lim_{b\to\infty}\Big(\int_0^b \frac{\ln x}{1+x^2}\,dx - \pi\int_0^b \frac{\ln x}{1+\pi^2x^2}\,dx\Big). \tag{15}$$

Consider now the integral

$$\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx.$$

Make the substitution $x = \pi u$, so that $dx = \pi du$ and

$$\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx = \pi \int_0^{b} \frac{\ln(\pi u)}{1+\pi^2u^2}\,du = \pi \int_0^{b} \frac{\ln u}{1+\pi^2u^2}\,du + \pi\ln\pi \int_0^{b} \frac{1}{1+\pi^2u^2}\,du, \tag{16}$$

where in the last step we used $\ln(\pi u) = \ln\pi + \ln u$. The last integral is easily computed:

$$\int_0^{b} \frac{1}{1+\pi^2u^2}\,du = \frac{1}{\pi}\arctan(\pi u)\Big|_0^b = \frac{1}{\pi}\arctan(\pi b) - \frac{1}{\pi}\arctan 0 = \frac{1}{\pi}\arctan(\pi b),$$

so that (16) becomes

$$\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx = \pi \int_0^{b} \frac{\ln u}{1+\pi^2u^2}\,du + \ln\pi\arctan(\pi b).$$

Notice that the variable u in the integral on the right hand side of this expression is just a dummy variable of integration, so we can relabel it as x, i.e.

$$\int_0^{b} \frac{\ln u}{1+\pi^2u^2}\,du = \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx,$$

and write

$$\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx = \pi \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx + \ln\pi\arctan(\pi b),$$

or equivalently,

$$\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx - \pi \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx = \ln\pi\arctan(\pi b).$$

Take the limit on both sides, i.e.,

$$\lim_{b\to\infty}\left(\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx - \pi \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx\right) = \ln\pi \lim_{b\to\infty}\arctan(\pi b). \tag{17}$$

When $b \to \infty$, $\pi b \to \infty$ as well, so we have

$$\lim_{b\to\infty}\int_0^{b} f(x)\,dx = \lim_{b\to\infty}\int_0^{\pi b} f(x)\,dx,$$

provided that the limit exists (see remark 1), and the left hand side of (17) becomes

$$\lim_{b\to\infty}\left(\int_0^{\pi b} \frac{\ln x}{1+x^2}\,dx - \pi \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx\right) = \lim_{b\to\infty}\left(\int_0^{b} \frac{\ln x}{1+x^2}\,dx - \pi \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx\right),$$

and therefore

$$\lim_{b\to\infty}\left(\int_0^{b} \frac{\ln x}{1+x^2}\,dx - \pi \int_0^{b} \frac{\ln x}{1+\pi^2x^2}\,dx\right) = \ln\pi \lim_{b\to\infty}\arctan(\pi b). \tag{18}$$

Using (18) into (15) gives

$$\lim_{b\to\infty}\lim_{a\to 0^+}\int_a^{b} \frac{\arctan(\pi x) - \arctan x}{x}\,dx = \ln\pi \lim_{b\to\infty}\arctan(\pi b).$$

Recalling that $\arctan x \to \frac{\pi}{2}$ when $x \to \infty$, we finally obtain

$$\lim_{b\to\infty} \lim_{a\to 0^+} \int_a^b \frac{\arctan(\pi x) - \arctan x}{x}\, dx = \frac{\pi}{2} \ln \pi.$$

Solution 2.
Make the substitution $u = \pi x$, so $du = \pi\, dx$. Then

$$\int_a^b \frac{\arctan(\pi x)}{x}\, dx = \int_{\pi a}^{\pi b} \frac{\arctan u}{u/\pi} \frac{1}{\pi}\, du = \int_{\pi a}^{\pi b} \frac{\arctan x}{x}\, dx,$$

where in the last step we used that u is a dummy variable of integration. Therefore

$$\int_a^b \frac{\arctan(\pi x) - \arctan x}{x}\, dx = \int_{\pi a}^{\pi b} \frac{\arctan x}{x}\, dx - \int_a^b \frac{\arctan x}{x}\, dx. \tag{19}$$

Breaking the first integral as

$$\int_{\pi a}^{\pi b} \frac{\arctan x}{x}\, dx = \int_{\pi a}^{b} \frac{\arctan x}{x}\, dx + \int_b^{\pi b} \frac{\arctan x}{x}\, dx,$$

makes (19) into

$$\int_a^b \frac{\arctan(\pi x) - \arctan x}{x}\, dx = \int_{\pi a}^{b} \frac{\arctan x}{x}\, dx + \int_b^{\pi b} \frac{\arctan x}{x}\, dx - \int_a^b \frac{\arctan x}{x}\, dx.$$

The first and third integrals can be combined as

$$\begin{aligned}\int_{\pi a}^{b} \frac{\arctan x}{x}\, dx - \int_a^b \frac{\arctan x}{x}\, dx &= -\int_b^{\pi a} \frac{\arctan x}{x}\, dx - \int_a^b \frac{\arctan x}{x}\, dx \\ &= -\int_a^{\pi a} \frac{\arctan x}{x}\, dx,\end{aligned}$$

so that

$$\int_a^b \frac{\arctan(\pi x) - \arctan x}{x}\, dx = \int_b^{\pi b} \frac{\arctan x}{x}\, dx - \int_a^{\pi a} \frac{\arctan x}{x}\, dx. \tag{20}$$

Since the first integral contains only b in the limits of integration and the second integral only a, the limit $b \to \infty$ applies exclusively to the first integral and the limit $a \to 0^+$ exclusively to the second one.

Let us compute

$$\lim_{a\to 0^+} \int_a^{\pi a} \frac{\arctan x}{x}\, dx = \lim_{a\to 0^+} \int_1^{\pi} \frac{\arctan(az)}{z}\, dz,$$

where we made the substitution $z = \frac{x}{a}$. Now notice that since $a > 0$, we have

$$0 \le \frac{\arctan(za)}{z} \le \frac{\arctan(\pi a)}{z},$$

for all $1 \le z \le \pi$. Therefore

$$0 \le \int_1^{\pi} \frac{\arctan(az)}{z}\, dz \le \int_1^{\pi} \frac{\arctan(\pi a)}{z}\, dz = \arctan(\pi a) \ln \pi.$$

Since $\arctan(\pi a) \to 0$ when $a \to 0^+$, the squeeze theorem gives

$$\lim_{a\to 0^+} \int_1^{\pi} \frac{\arctan(az)}{z} = 0,$$

or yet

$$\lim_{a\to 0^+}\int_a^{\pi a}\frac{\arctan x}{x}\,dx = 0. \tag{21}$$

Now consider

$$\lim_{b\to\infty}\int_b^{\pi b}\frac{\arctan x}{x}\,dx = \lim_{b\to\infty}\int_1^{\pi}\frac{\arctan(bw)}{w}\,dw, \tag{22}$$

where we made the substitution $w = \frac{x}{b}$. Since for $b > 0$ we have

$$\frac{\arctan b}{w} \leq \frac{\arctan(bw)}{w} \leq \frac{\arctan(b\pi)}{w},$$

for $1 \leq w \leq \pi$, we conclude that

$$\int_1^{\pi}\frac{\arctan b}{w}\,dw \leq \int_1^{\pi}\frac{\arctan(bw)}{w}\,dw \leq \int_1^{\pi}\frac{\arctan(b\pi)}{w}\,dw.$$

Computing the integrals on the left and on the right we obtain

$$\arctan b \ln \pi \leq \int_1^{\pi}\frac{\arctan(bw)}{w}\,dw \leq \arctan(\pi b)\ln\pi,$$

and therefore, applying the squeeze theorem,

$$\lim_{b\to\infty}\int_1^{\pi}\frac{\arctan(bw)}{w}\,dw = \frac{\pi}{2}\ln\pi,$$

since

$$\lim_{b\to\infty}\arctan b = \frac{\pi}{2}, \quad \text{and} \quad \lim_{b\to\infty}\arctan(\pi b) = \frac{\pi}{2}.$$

Therefore (22) becomes

$$\lim_{b\to\infty}\int_b^{\pi b}\frac{\arctan x}{x}\,dx = \ln\pi\frac{\pi}{2}. \tag{23}$$

Using (21) and (23) into (20) gives

$$\lim_{b\to\infty}\lim_{a\to 0^+}\int_a^b\frac{\arctan(\pi x)-\arctan x}{x}\,dx = \ln\pi\frac{\pi}{2},$$

as desired.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`