MAT 155B - FALL 12 - EXAMPLES SECTION 7.8

Question 1. Evaluate the integrals:

$$(a)\int_0^\infty (1-x)e^{-x}\,dx.$$
$$(b)\int_\varepsilon^\infty \frac{1}{\sqrt{x}(1+x)}\,dx,\ \text{where } \varepsilon>0.$$

Question 2. Identify whether the integral

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$$

is convergent. If it converges, evaluate it.

SOLUTIONS.

1a. We have

$$\int_0^\infty (1-x)e^{-x}\,dx = \lim_{t\to\infty}\int_0^t (1-x)e^{-x}\,dx.$$

Use integration by parts with $u=(1-x)$, $dv=e^{-x}dx$, so that $du=-dx$, $v=-e^{-x}$ and

$$\begin{aligned}\lim_{t\to\infty}\int_0^t (1-x)e^{-x}\,dx &= \lim_{t\to\infty}\Big(-(1-x)e^{-x}\Big|_0^t - \int_0^t e^{-x}\,dx\Big)\\ &= \lim_{t\to\infty}\Big(-(1-x)e^{-x}\Big|_0^t + e^{-x}\Big|_0^t\Big)\\ &= \lim_{t\to\infty}\Big(-(1-t)e^{-t}+1+e^{-t}-1\Big)\\ &= -\lim_{t\to\infty}(1-t)e^{-t}+\lim_{t\to\infty}e^{-t}\\ &= 0\end{aligned}$$

1b. Write

$$\int_\varepsilon^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to\infty}\int_\varepsilon^t \frac{1}{\sqrt{x}(1+x)}\,dx.$$

Let us compute the indefinite integral first. Write $x=(\sqrt{x})^2$ so

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{1}{1+(\sqrt{x})^2}\frac{1}{\sqrt{x}}\,dx,$$

and make the u-substitution $u=\sqrt{x}$, so $du=\frac{1}{2\sqrt{x}}\,dx$ and

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int \frac{1}{1+u^2}\,du = 2\arctan u = 2\arctan\sqrt{x}.$$

Hence

$$\int_\varepsilon^t \frac{1}{\sqrt{x}(1+x)}\,dx = 2\arctan\sqrt{t} - 2\arctan\sqrt{\varepsilon}.$$

Taking the limit

$$\int_\varepsilon^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to\infty} 2\arctan\sqrt{t} - 2\arctan\sqrt{\varepsilon}$$
$$= 2\frac{\pi}{2} - 2\arctan\sqrt{\varepsilon} = \pi - 2\arctan\sqrt{\varepsilon},$$

where we have used that $\arctan x \to \frac{\pi}{2}$ when $x \to \infty$.

2. Notice that $e^{2x}+3 > e^{2x}$, so $\frac{1}{e^{2x}+3} < \frac{1}{e^{2x}}$. We then obtain

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx < \int_0^\infty \frac{e^x}{e^{2x}}\,dx = \int_0^\infty e^{-x}\,dx.$$

This last integral is easily computed:

$$\int_0^\infty e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_0^t = -\lim_{t\to\infty}(e^{-t}-1) = 1.$$

So the original integral is convergent by the comparison test. Now, for computing it, do the substitution $u = e^x$, so $du = e^x\,dx$. Then

$$\int_0^t \frac{e^x}{e^{2x}+3}dx = \int_0^t \frac{e^x}{(e^x)^2+3}dx = \int_1^{e^t}\frac{du}{u^2+3} = \frac{1}{3}\int_1^{e^t}\frac{du}{\frac{u^2}{3}+1}.$$

Now write $\frac{u^2}{3}$ as $\left(\frac{u}{\sqrt{3}}\right)^2$, make the substitution $\frac{u}{\sqrt{3}} = v$, so that $du = \sqrt{3}dv$ and obtain

$$\frac{1}{3}\int_1^{e^t}\frac{du}{\frac{u^2}{3}+1} = \frac{\sqrt{3}}{3}\int_{\frac{1}{\sqrt{3}}}^{\frac{e^t}{\sqrt{3}}}\frac{dv}{v^2+1} = \frac{\sqrt{3}}{3}\arctan v\Big|_{\frac{1}{\sqrt{3}}}^{\frac{e^t}{\sqrt{3}}}$$
$$= \frac{\sqrt{3}}{3}(\arctan\frac{e^t}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}}).$$

Recall that $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. Now we need to take the limit $t\to\infty$,

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+3}dx = \lim_{t\to\infty}\frac{\sqrt{3}}{3}(\arctan\frac{e^t}{\sqrt{3}} - \frac{\pi}{6})$$
$$= \frac{\sqrt{3}}{3}(\lim_{t\to\infty}\arctan\frac{e^t}{\sqrt{3}} - \frac{\pi}{6}) = \frac{\sqrt{3}}{3}(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{\sqrt{3}\pi}{9}.$$

Here we used that $\lim_{t\to\infty}\frac{e^t}{\sqrt{3}} = \infty$ and that the tangent of some angle θ approaches infinity when the angle approaches $\frac{\pi}{2}$, therefore $\lim_{t\to\infty}\arctan\frac{e^t}{\sqrt{3}} = \frac{\pi}{2}$.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`