MAT 155B - FALL 12 - EXAMPLES SECTION 7.4

Question. Integrate:

$$(a) \int \frac{x^2+2x-1}{x^3-x}\,dx$$
$$(b) \int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$$

Solutions.

a. Write

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1},$$

so that

$$\begin{aligned} x^2+2x-1 =& A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \\ &= Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx \\ &= (A+B+C)x^2 + (-B+C)x - A \end{aligned}$$

Equating the coefficients of x^2, x and the constant term on each side yields

$$\begin{cases} A+B+C=1 \\ -B+C=2 \\ -A=-1 \end{cases}$$

Solving the system we find

$$A=1,\ B=-1,\ C=1.$$

We therefore have

$$\begin{aligned} \int \frac{x^2+2x-1}{x^3-x}dx &= \int \Big(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\Big)\,dx \\ &= \ln|x| - \ln|x+1| + \ln|x-1| + C. \end{aligned}$$

b. Notice that x^2+2x+2 is irreducible since $2^2 - 4\times 1\times 2 < 0$ (recall that ax^2+bx+c is irreducible whenever $b^2-4ac<0$).

Write

$$\begin{aligned} \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} &= \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2} \\ &= \frac{(Ax+B)(x^2+2x+2)+(Cx+D)}{(x^2+2x+2)^2}. \end{aligned}$$

Then

$$\begin{aligned} x^3+2x^2+3x-2 &= (Ax+B)(x^2+2x+2) + (Cx+D) \\ &= Ax^3 + (2A+B)x^2 + (2A+2B+C)x + (2B+D), \end{aligned}$$

which gives the system

$$\begin{cases} A = 1 \\ 2A + B = 2 \\ 2A + 2B + 2C = 3 \\ 2B + D = -2 \end{cases}$$

whose solution is

$$A = 1,\ B = 0,\ C = 1,\ D = -2.$$

Therefore

$$\begin{aligned} \int \frac{x^3 + 2x^2 + 3x - 2}{(x^2 + 2x + 2)^2}\, dx &= \int \Big(\frac{x}{x^2 + 2x + 2} + \frac{x - 2}{(x^2 + 2x + 2)^2} \Big)\, dx \\ &= \int \frac{x}{x^2 + 2x + 2} dx + \int \frac{x - 2}{(x^2 + 2x + 2)^2}\, dx \end{aligned} \tag{1}$$

Write the first integral as

$$\begin{aligned} \int \frac{x}{x^2 + 2x + 2} dx &= \int \frac{x + 1 - 1}{x^2 + 2x + 2} dx \\ &= \int \frac{x + 1}{x^2 + 2x + 2} dx - \int \frac{1}{x^2 + 2x + 2} dx, \end{aligned}$$

and the second one as

$$\begin{aligned} \int \frac{x - 2}{(x^2 + 2x + 2)^2}\, dx &= \int \frac{x - 2 + 1 - 1}{(x^2 + 2x + 2)^2}\, dx \\ &= \int \frac{x + 1}{(x^2 + 2x + 2)^2}\, dx - 3 \int \frac{1}{(x^2 + 2x + 2)^2}\, dx, \end{aligned}$$

so that (1) becomes

$$\begin{aligned} \int \frac{x^3 + 2x^2 + 3x - 2}{(x^2 + 2x + 2)^2}\, dx = &\int \frac{x + 1}{x^2 + 2x + 2}\, dx - \int \frac{1}{x^2 + 2x + 2}\, dx \\ &+ \int \frac{x + 1}{(x^2 + 2x + 2)^2}\, dx - 3 \int \frac{1}{(x^2 + 2x + 2)^2}\, dx. \end{aligned} \tag{2}$$

Let us compute each integral in (2) separately.

For the first integral on the right hand side of (2), make the u-substitution $u = x^2 + 2x + 2$, so that $du = 2(x + 1)dx$ and then

$$\int \frac{x + 1}{x^2 + 2x + 2}\, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| = \frac{1}{2} \ln |x^2 + 2x + 2| \tag{3}$$

(we are going to add the constant of integration at the end).

For the second integral on the right hand side of (2), notice that $x^2 + 2x + 2 = (x + 1)^2 + 1$, so that

$$\int \frac{1}{x^2 + 2x + 2}\, dx = \int \frac{1}{(x + 1)^2 + 1}\, dx = \arctan(x + 1) \tag{4}$$

For the third integral on the right hand side of (2), make the u-substitution $u = x^2 + 2x + 2$, so that $du = 2(x + 1)dx$ and then

$$\int \frac{x + 1}{(x^2 + 2x + 2)^2}\, dx = \frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2u} = -\frac{1}{2(x^2 + 2x + 2)} \tag{5}$$

Finally, for the fourth integral on the right hand side of (2), write $x^2+2x+2=(x+1)^2+1$, so that

$$\int \frac{1}{(x^2+2x+2)^2}\,dx = \int \frac{1}{[(x+1)^2+1]^2}\,dx.$$

Make the substitution $x+1=\tan\theta$, so that $dx=\sec^2\theta\,d\theta$ and then

$$\begin{aligned}\int \frac{1}{[(x+1)^2+1]^2}\,dx &= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta \\ &= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta.\end{aligned}$$

Recalling that $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$,

$$= \frac{1}{2}\int (1+\cos(2\theta))\,d\theta = \frac{1}{2}(\theta + \frac{\sin(2\theta)}{2}) = \frac{1}{2}(\theta+\sin\theta\cos\theta),$$

where we have used $\sin(2\theta)=2\sin\theta\cos\theta$. Since $\tan\theta = x+1 = \frac{x+1}{1}$, we get

$$\begin{aligned}\theta &= \arctan(x+1) \\ \sin\theta &= \frac{x+1}{\sqrt{x^2+2x+2}} \\ \cos\theta &= \frac{1}{\sqrt{x^2+2x+2}},\end{aligned}$$

and therefore

$$\int \frac{1}{(x^2+2x+2)^2}\,dx = \frac{1}{2}(\theta+\sin\theta\cos\theta) = \frac{1}{2}\arctan(x+1) + \frac{1}{2}\frac{x+1}{x^2+2x+2}. \tag{6}$$

Using (3), (4), (5), and (6) into (2) finally yields, after some simplifications,

$$\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \frac{1}{2}\ln(x^2+2x+2) - \frac{5}{2}\arctan(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`