MAT 155B - FALL 12 - EXAMPLES SECTION 7.1

Question. Compute the integral

$$\int (\sin^{-1} x)^2 \, dx$$

Solution. Write

$$\int (\sin^{-1} x)^2 \, dx = \int \sin^{-1} x \sin^{-1} x \, dx$$

Use integration by parts. Let

$$u = \sin^{-1} x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} \, dx$$
$$dv = \sin^{-1} x \, dx \Rightarrow v = \int \sin^{-1} x dx$$

To find v, do another integration by parts:

$$U = \sin^{-1} x \Rightarrow dU = \frac{1}{\sqrt{1-x^2}} \, dx$$
$$dV = dx \Rightarrow V = x$$

Then

$$v = \int \sin^{-1} x \, dx = UV - \int V \, dU = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \sin^{-1} x + \sqrt{1-x^2}$$

Hence $v = x \sin^{-1} x + \sqrt{1-x^2}$ and

$$\begin{aligned} \int (\sin^{-1} x)^2 \, dx &= uv - \int v \, du \\ &= \sin^{-1} x (x \sin^{-1} x + \sqrt{1-x^2}) - \int \left(x \sin^{-1} x + \sqrt{1-x^2} \right) \frac{1}{\sqrt{1-x^2}} \, dx \\ &= x (\sin^{-1} x)^2 + \sqrt{1-x^2} \sin^{-1} x - x - \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx \qquad (*) \end{aligned}$$

For the last integral make the substitution

$$z = \sin^{-1} x \Rightarrow x = \sin z$$
$$dz = \frac{1}{\sqrt{1-x^2}} \, dx$$

Then

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx = \int z \sin z dz$$

This integral can also be done by integration by parts. Put

$$\mathcal{U} = z \Rightarrow d\mathcal{U} = dz$$
$$d\mathcal{V} = \sin z \, dz \Rightarrow \mathcal{V} = -\cos z$$

Then

$$\int z \sin z dz = \mathcal{U}\mathcal{V} - \int \mathcal{V} \, d\mathcal{U} = -z \cos z + \int \cos z \, dz = -z \cos z + \sin z$$

Recalling that $z = \sin^{-1} x$ and using the identity $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ (we showed this identity in class), we have

$$\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} \, dx = -\sqrt{1 - x^2} \sin^{-1} x + x$$

Using this into $(*)$ finally gives

$$\int (\sin^{-1} x)^2 \, dx = x(\sin^{-1} x)^2 + 2\sqrt{1 - x^2} \sin^{-1} x - 2x + C$$