MAT 155B - FALL 12 - EXAMPLES SECTION 6.8

Compute the following limits:

(a) $\lim_{x\to 0} \dfrac{\sin(7x)}{\sin(12x)}$

(b) $\lim_{x\to 0} \dfrac{e^{2x}-1}{x^2}$

(c) $\lim_{x\to\infty} e^{-x}\ln x$

(d) $\lim_{x\to \frac{\pi}{4}^-} (1-\tan x)\sec 2x$

(e) $\lim_{x\to\infty} \left(1+\dfrac{\pi}{x}\right)^x$

Solutions.

(a) $\lim_{x\to 0} \dfrac{\sin(7x)}{\sin(12x)} = \dfrac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \dfrac{(\sin(7x))'}{(\sin(12x))'} = \lim_{x\to 0} \dfrac{7\cos(7x)}{12\cos(12x)} = \dfrac{7}{12}$

(b) $\lim_{x\to 0} \dfrac{e^{2x}-1}{x^2} = \dfrac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \dfrac{(e^{2x}-1)'}{(x^2)'} = \lim_{x\to 0} \dfrac{2e^{2x}}{2x} = \lim_{x\to 0} \dfrac{e^{2x}}{x}$

The top approaches 1 when $x\to 0$. But $\frac{1}{x} \to +\infty$ when $x\to 0^+$ and $\frac{1}{x}\to -\infty$ when $x\to 0^-$, hence the limit does not exist.

Remark 1. Notice that if we wanted to compute

$$\lim_{x\to 0^+} \frac{e^{2x}-1}{x^2},$$

then the answer would be $+\infty$, whereas the answer would be $-\infty$ for

$$\lim_{x\to 0^-} \frac{e^{2x}-1}{x^2},$$

(c) $\lim_{x\to\infty} e^{-x}\ln x = 0\cdot\infty$

Here we cannot apply L'Hospital rule directly since the indeterminacy is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. But we can write

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{(\ln x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

(d) $\lim\limits_{x\to\frac{\pi}{4}^-} (1-\tan x)\sec 2x = 0\cdot\infty$

Again we cannot use L'Hospital rule directly, but we can write

$$\lim_{x\to\frac{\pi}{4}^-} (1-\tan x)\sec 2x = \lim_{x\to\frac{\pi}{4}^-} \frac{1-\tan x}{\cos 2x} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to\frac{\pi}{4}^-} \frac{(1-\tan x)'}{(\cos 2x)'}$$

$$= \lim_{x\to\frac{\pi}{4}^-} \frac{-\sec^2 x}{-2\sin 2x} = \frac{\sec^2 \frac{\pi}{4}}{2\sin\frac{\pi}{2}} = \frac{2}{2\cdot 1} = 1$$

(e) $\lim\limits_{x\to\infty} \left(1+\frac{\pi}{x}\right)^x = 1^\infty$

Once more, L'Hospital rule cannot be directly applied, so we need to rework the expression. Let

$$y = \left(1+\frac{\pi}{x}\right)^x$$

Then

$$\ln y = \ln\left(1+\frac{\pi}{x}\right)^x = x\ln\left(1+\frac{\pi}{x}\right) = \frac{\ln\left(1+\frac{\pi}{x}\right)}{\frac{1}{x}}$$

So

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{\pi}{x}\right)}{\frac{1}{x}} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\ln\left(1+\frac{\pi}{x}\right)\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{\pi}{x}}\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}} = \pi$$

Then

$$\lim_{x\to\infty} \left(1+\frac{\pi}{x}\right)^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^\pi$$

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`