**MAT 155B - FALL 12 - EXAMPLES SECTION 6.4***

**1.** If $f(3) = 4$, $f(12) = 17$ and $f$ is an exponential function, find $f'(x)$.

**2.** Find the derivative of the given functions.

(a) $y = \ln x \, 2^x$ (b) $f(t) = t^e \, e^t$

**3.** The half-life of an element is defined as the time necessary for a given amount (of that element) to decay to half of its value. The half-life of the Plutonium-239, ${}^{239}Pu$, is 24 110 years. Assuming that the process has exponential behavior:

(a) Find a formula for the amount of ${}^{239}Pu$ remaining after $t$ years.

(b) How long does it take for a sample of ${}^{239}Pu$ to decay to 30% of its initial value?

(c) If 0.05 grams of ${}^{239}Pu$ are initially present, how much will remain after five thousand years?

**Solutions.**

**1.** Write $f(x) = Ca^x$. We are given $f(3) = Ca^3 = 4$ and $f(12) = Ca^{12} = 17$. Thus we obtain the system

$$\begin{cases} Ca^3 = 4, \\ Ca^{12} = 17. \end{cases}$$

Dividing both equations yields

$$\frac{Ca^{12}}{Ca^3} = a^9 = \frac{17}{4} \Rightarrow a = \left(\frac{17}{4}\right)^{\frac{1}{9}}.$$

Using this into $Ca^3 = 4$ gives

$$Ca^3 = C\left(\frac{17}{4}\right)^{\frac{3}{9}} = 4 \Rightarrow C = \frac{4^{\frac{4}{3}}}{17^{\frac{1}{3}}}.$$

Hence

$$f(x) = Ca^x = \frac{4^{\frac{4}{3}}}{17^{\frac{1}{3}}}\left(\frac{17}{4}\right)^{\frac{x}{9}},$$

from which we conclude, using that $(a^x)' = \ln a \, a^x$, that

$$f'(x) = \frac{1}{9}\frac{4^{\frac{4}{3}}}{17^{\frac{1}{3}}} \ln\frac{17}{4}\left(\frac{17}{4}\right)^{\frac{x}{9}}.$$

**2a.** Use the product rule, $(\ln x)' = \frac{1}{x}$ and $(a^x)' = \ln a \, a^x$ to find

$$y' = \frac{2^x}{x} + \ln x \ln 2 \, 2^x.$$

**2b.** Use the product rule, the power rule and $(e^x)' = e^x$ to find

$$f'(t) = (t^e)'\, e^t + t^e (e^t)' = e t^{e-1} e^t + t^e e^t = (\frac{e}{t} + 1) t^e e^t.$$

**Remark 1.** Notice that $t^e$ is a power function since $e$ is simply a real number. That's why $(t^e)' = et^{e-1}$.

**3a.** Denote by $A(t)$ the amount of ${}^{239}Pu$ at time $t$. The assumption is that

$$A(t) = Ca^t.$$

Since $A(0) = C$, the constant $C$ is simply the amount of ${}^{239}Pu$ at time zero, hence we denote $C = A_0$. Write then $A(t) = A_0 a^t$.

After 24 110 years there will be half of the substance left, i.e., $\frac{A_0}{2}$. Thus

$$A(24\,110) = A_0 a^{24\,110} = \frac{A_0}{2} \Rightarrow a^{24\,110} = \frac{1}{2} \Rightarrow \ln a = \frac{1}{24\,110} \ln \frac{1}{2},$$

or

$$a = e^{\frac{1}{24\,110} \ln \frac{1}{2}}.$$

Notice that $A_0$ cancels out, so we don't need to know the initial amount in order to find $a$. We can simplify things a little by noticing that

$$\frac{1}{24\,110} \ln \frac{1}{2} = \ln 2^{-\frac{1}{24\,110}},$$

and then

$$a = e^{\frac{1}{24\,110} \ln \frac{1}{2}} = e^{\ln 2^{-\frac{1}{24\,110}}} = 2^{-\frac{1}{24\,110}}.$$

This leads to

$$A(t) = A_0 2^{-\frac{t}{24\,110}},$$

which is the desired formula.

**3b.** We want $t$ such that $A(t) = 0.3 A_0$, so

$$A(t) = 0.3 A_0 = A_0 2^{-\frac{t}{24\,110}} \Rightarrow 0.3 = 2^{-\frac{t}{24\,110}} \Rightarrow t = -24\,110 \log_2 0.3.$$

Once more, the value of $A_0$ is not necessary to solve the problem. Notice that the value of $t$ found above is positive since $\log_2 0.3 < 0$.

**3c.** Use the formula $A(t) = A_0 2^{-\frac{t}{24\,110}}$ with $A_0 = 0.05$ and $t = 5\,000$ to find

$$A = 0.05 \times 2^{-\frac{5\,000}{24\,110}} \text{ grams.}$$

*URL*: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`