MAT 155B - FALL 12 - EXAMPLES SECTION 11.9

Question 1: Find a power series representation for the functions below.

$$(a) f(x) = \frac{x}{(1+4x)^2} \qquad (b) f(x) = \ln\Big(\frac{1+x}{1-x}\Big)$$

Question 2: Evaluate the integral as a power series.

$$(a) \int \frac{\ln(1-t)}{t} dt \qquad (b) \int_0^{0.1} x \arctan(3x) dx \quad \text{up to six decimal places}$$

Solutions.

(1a) When you want to find a power series representation of functions involving $\frac{1}{(a+bx)^\ell}$, with a, b and ℓ constants, the idea is to use derivatives and the geometric series. We can at first ignore the x term on the numerator, since it can be multiplied later on, i.e., after obtaining the power series for $\frac{1}{(1+4x)^2}$.

First notice that

$$\frac{1}{(1+4x)^2} = -\frac{1}{4}\frac{d}{dx}\Big(\frac{1}{1+4x}\Big).$$

How do we know the correct constant in front of the derivative, i.e., how did we figure out the factor $-\frac{1}{4}$? To see why the above equality is true, start with $\frac{1}{1+4x}$ and then take derivatives

$$\frac{d}{dx}\Big(\frac{1}{1+4x}\Big) = -\frac{4}{(1+4x)^2} \Rightarrow \frac{1}{(1+4x)^2} = -\frac{1}{4}\frac{d}{dx}\Big(\frac{1}{1+4x}\Big).$$

Now we use the geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ with $r = -4x$:

$$\frac{1}{1+4x} = \sum_{n=0}^{\infty} (-1)^n 4^n x^n,$$

which converges for $|x| < \frac{1}{4}$ since we must have $|r| < 1$. Therefore

$$\frac{1}{(1+4x)^2} = -\frac{1}{4}\frac{d}{dx}\Big(\frac{1}{1+4x}\Big) = -\frac{1}{4}\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n 4^n x^n = -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n 4^n \frac{d}{dx}x^n$$
$$= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n 4^n n x^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1}4^{n-1}nx^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1}4^{n-1}nx^{n-1}.$$

In the last step we started the sum at $n = 1$ because the $n = 0$ term vanishes. Now we can multiply by x:

$$\frac{x}{(1+4x)^2} = x\sum_{n=1}^{\infty}(-1)^{n+1}4^{n-1}nx^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1}4^{n-1}nx^n.$$

Remark: Suppose you were looking for a power series for $\frac{1}{(1-8x)^3}$. Then you would differentiate $\frac{1}{1-8x}$ twice:

$$\frac{d}{dx}\Big(\frac{1}{1-8x}\Big) = \frac{8}{(1-8x)^2},$$
$$\frac{d^2}{dx^2}\Big(\frac{1}{1-8x}\Big) = \frac{d}{dx}\Big(\frac{8}{(1-8x)^2}\Big) = \frac{128}{(1-8x)^3}.$$

Therefore

$$\frac{1}{(1-8x)^3} = \frac{1}{128}\frac{d^2}{dx^2}\Big(\frac{1}{1-8x}\Big).$$

The term in parenthesis on the right hand side can then be expanded as a geometric series, and after that you can take the derivative $\frac{d^2}{dx^2}$ as in the example above, except that in this case you would have to differentiate twice.

(1b) Use the property

$$\ln\Big(\frac{1+x}{1-x}\Big) = \ln(1+x) - \ln(1-x).$$

We will see in class that

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \ |x| < 1.$$

Replace x by $-x$ in the above formula to find

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \ |x| < 1.$$

Putting these two formulas together we get

$$\ln\Big(\frac{1+x}{1-x}\Big) = \ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + \sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\Big((-1)^{n-1}\frac{x^n}{n} + \frac{x^n}{n}\Big).$$

Now notice that the term in parenthesis equals zero if n is even and $\frac{2x^n}{n}$ if n is odd. So

$$\ln\Big(\frac{1+x}{1-x}\Big) = 2\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}\frac{x^n}{n}.$$

Summing only over n odd is the same as summing over all n with n replaced by $2n+1$, so

$$\ln\Big(\frac{1+x}{1-x}\Big) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}.$$

(2a) Again, use the formula

$$\ln(1-t) = -\sum_{n=1}^{\infty}\frac{t^n}{n}, \ |t| < 1.$$

Then

$$\frac{\ln(1-t)}{t} = -\frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{n} = -\sum_{n=1}^{\infty}\frac{t^{n-1}}{n}.$$

Integrating:

$$\int \frac{\ln(1-t)}{t} dt = -\int \Big(\sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \Big) dt = -\sum_{n=1}^{\infty} \int \frac{t^{n-1}}{n} dt.$$

Recalling that

$$\int t^{n-1} dt = \frac{t^n}{n},$$

we get

$$\int \frac{\ln(1-t)}{t} dt = -\sum_{n=1}^{\infty} \int \frac{t^{n-1}}{n} dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2},$$

where C is a constant of integration.

(2b) We will see in class that

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},$$

for $|x| \leq 1$. It follows that

$$\arctan(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{2n+1},$$

and this converges for $|x| \leq \frac{1}{3}$. Hence

$$x \arctan(3x) = x \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+2}}{2n+1}.$$

Integrate to get:

$$\int_0^{0.1} x \arctan(3x) dx = \int_0^{0.1} \Big(\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+2}}{2n+1} \Big) dx = \sum_{n=0}^{\infty} \int_0^{0.1} \frac{(-1)^n 3^{2n+1} x^{2n+2}}{2n+1} dx.$$

Since

$$\int_0^{0.1} x^{2n+2} dx = \frac{x^{2n+3}}{2n+3}\Big|_0^{0.1} = \frac{(0.1)^{2n+3}}{2n+3} = \frac{1}{(2n+3)10^{2n+3}},$$

we obtain

$$\int_0^{0.1} x \arctan(3x) dx = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)(2n+3)10^{2n+3}}.$$

Notice that this is an alternating series with $b_n = \frac{3^{2n+1}}{(2n+1)(2n+3)10^{2n+3}}$. We can now use remainder estimates for the alternating series to get:

$$|R_n| \leq b_{n+1} = \frac{3^{2n+3}}{(2n+3)(2n+5)10^{2n+5}}.$$

We want this to be of the order 10^{-6}. Since there is a n appearing in several places, it's cumbersome to solve $b_{n+1} \leq 10^{-6}$ directly for n. But noticing that

$$\frac{3^{2n+3}}{(2n+3)(2n+5)10^{2n+5}} \leq \frac{3^{2n+3}}{10^{2n+5}} \leq \frac{3^{2n+5}}{10^{2n+5}} = \Big(\frac{3}{10}\Big)^{2n+5},$$

we see that it is enough to have

$$\left(\frac{3}{10}\right)^{2n+5} \leq 10^{-6}.$$

Playing with n values we see that for $n = 4$ we get

$$\left(\frac{3}{10}\right)^{13} = 0.00000016.$$

Hence we can sum up to $n = 4$ to obtain

$$\int_0^{0.1} x \arctan(3x) dx \approx \sum_{n=0}^{4} \frac{(-1)^n 3^{2n+1}}{(2n+1)(2n+3)10^{2n+3}} = 0.000982662,$$

where to obtain the numerical value a calculator has been used.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`