MAT 155B - FALL 12 - EXAMPLES SECTION 11.5

Question 1. Determine whether the given series converges. If it does, how many terms do we need to use to obtain an approximation to its sum up to three decimal digits?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n^2+n}.$$

SOLUTION.

This is a alternating series, so we want to use the alternating series test. First, we identify

$$b_n = \frac{n+3}{n^2+n}.$$

Clearly, $b_n \to 0$ when $n \to \infty$. Next, we need to test if $b_{n+1} \leq b_n$, in other words, if b_n is a decreasing sequence. For this, we consider the function

$$f(x) = \frac{x+3}{x^2+x},$$

and check whether $f(x)$ is decreasing; if it is, so will be b_n. Recall that a function is decreasing when its derivative is negative, so compute:

$$\begin{aligned} f'(x) &= \frac{(x^2+x)(-1)-(x+3)(2x+1)}{(x^2+x)^2} \\ &= \frac{-x^2-6x-3}{(x^2+x)^2} \\ &= -\frac{x^2+6x+3}{(x^2+x)^2}. \end{aligned}$$

We want $f'(x) \leq 0$. Now, recall that whenever we consider this trick of replacing b_n by $f(x)$ and compute $f'(x)$, it is **enough to test the condition $f'(x) \leq 0$ for positive values of** x, since we are really interested in b_n, and $n \geq 0$. But since for $x \geq 0$ we have

$$\frac{x^2+6x+3}{(x^2+x)^2} \geq 0,$$

we conclude that

$$f'(x) = -\frac{x^2+6x+3}{(x^2+x)^2} \leq 0.$$

Thus, $b_{n+1} \leq b_n$ and the series converges by the alternating series test.

To estimate the error, we use

$$|s - s_N| \leq b_{N+1}.$$

We want the error to be less than or equal to 0.001. Since

$$b_{N+1} = \frac{N+1+3}{(N+1)^2+N+1},$$

we want

$$\frac{N+1+3}{(N+1)^2+N+1} \leq 0.001 = \frac{1}{1000}.$$

We need to solve this inequality for N. The inequality is equivalent to

$$N^2 - 997N - 3998 \geq 0.$$

Using the quadratic formula we find

$$N \geq \frac{997 + \sqrt{(997)^2 + 4 \cdot 3998}}{2} = \frac{997 + \sqrt{1010001}}{2} \approx 1000 \text{ terms} .$$

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`