Question 1a Write $y = h(x) = 17 + \sqrt[5]{\frac{x}{e}}$ and solve for x:

$y - 17 = \sqrt[5]{\frac{x}{e}} \Rightarrow (y-17)^5 = \frac{x}{e} \Rightarrow x = e(y-17)^5$

So $h^{-1}(x) = e(x-17)^5$

Question 1b $y = \ln(\tan \frac{1}{x})$. Solve for x:

$y = \ln(\tan \frac{1}{x}) \Rightarrow e^y = \tan \frac{1}{x} \Rightarrow \arctan e^y = \frac{1}{x}$

$\Rightarrow x = \frac{1}{\arctan e^y}$. Hence $y^{-1}(x) = \frac{1}{\arctan e^y}$

Question 1c $y = \frac{e^x}{1+2e^x}$. Again, solve for x:

$(1+2e^x)y = e^x \Rightarrow y + 2e^x y = e^x \Rightarrow y = e^x - 2e^x y$

$\Rightarrow y = (1-2y)e^x \Rightarrow e^x = \frac{y}{1-2y} \Rightarrow x = \ln \frac{y}{1-2y}$

Hence $y^{-1}(x) = \ln\left(\frac{x}{1-2x}\right)$

Question 2a Use the chain rule:

$$y' = \left(\frac{1}{\ln\cos e^x}\right)' = -\frac{1}{(\ln\cos e^x)^2}\cdot(\ln\cos e^x)' =$$

$$= -\frac{1}{(\ln\cos e^x)^2}\frac{1}{\cos e^x}\cdot(\cos e^x)' = \frac{\sin(e^x)\cdot e^x}{(\ln\cos e^x)^2\cdot\cos e^x}$$

Question 2b Use the chain rule and remember that $(\arctan x)' = \frac{1}{1+x^2}$. So $y' = \left(\arctan\sqrt{\frac{1-x}{1+x}}\right)'$

$$= \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\cdot\left(\sqrt{\frac{1-x}{1+x}}\right)' = \frac{1+x}{1+x+1-x}\left(\sqrt{\frac{1-x}{1+x}}\right)'$$

$$= \frac{(1+x)}{2}\cdot\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}\cdot\overbrace{\left(\frac{1-x}{1+x}\right)'}^{\text{use quotient rule}} = \frac{1}{4}\frac{(1+x)^{3/2}}{\sqrt{1-x}}\cdot\frac{(1+x)\cdot(1-x)'-(1-x)(1+x)'}{(1+x)^2}$$

$$= \frac{1}{4}\frac{(1+x)^{3/2}}{\sqrt{1-x}}\cdot\frac{-(1+x)-(1-x)}{(1+x)^2} = \frac{1}{4}\frac{(1+x)^{3/2}}{\sqrt{1-x}}\cdot\frac{(-2)}{(1+x)^2}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{(1-x)\cdot(1+x)}} = -\frac{1}{2\sqrt{1-x^2}}$$

Question 2c Use the chain rule and recall that

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$y' = \left(\arccos\left(\frac{2+3\cos x}{3+2\cos x}\right)\right)' = \frac{-1}{\sqrt{1-\left(\frac{2+3\cos x}{3+2\cos x}\right)^2}} \cdot \overbrace{\left(\frac{2+3\cos x}{3+2\cos x}\right)'}^{\text{use quotient rule}}$$

$$= -\frac{1}{\sqrt{\frac{(3+2\cos x)^2-(2+3\cos x)^2}{(3+2\cos x)^2}}} \cdot \frac{(3+2\cos x)\cdot(-3\sin x)-(2+3\cos x)(-2\sin x)}{(3+2\cos x)^2}$$

$$= \frac{-\sqrt{(3+2\cos x)^2}}{\sqrt{(3+2\cos x)^2-(2+3\cos x)^2}} \cdot \frac{-9\sin x - 6\sin x\cos x + 4\sin x + 6\sin x\cos x}{(3+2\cos x)^2}$$

$$= \frac{5\sin x}{|3+2\cos x|\sqrt{(3+2\cos x)^2-(2+3\cos x)^2}}$$

<u>Question 2d</u> Recalling that $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$:

$$f'(\theta) = (\theta \ln(\arctan\theta))' - (\arcsin\sqrt{\sin\theta})'$$

$$= \ln(\arctan\theta) + \frac{\theta}{\arctan\theta}(\arctan\theta)' - \frac{1}{\sqrt{1-(\sqrt{\sin\theta})^2}} \cdot (\sqrt{\sin\theta})'$$

$$= \ln(\arctan\theta) + \frac{\theta}{\arctan\theta(1+\theta^2)} - \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{(\sin\theta)'}{2\sqrt{\sin\theta}}$$

$$= \ln(\arctan\theta) + \frac{\theta}{\arctan\theta(1+\theta^2)} - \frac{\cos\theta}{2\sqrt{\sin\theta(1-\sin\theta)}}$$

<u>Question 3a</u> Write

$$\int \frac{7}{11+2x^2}\,dx = \frac{7}{11}\int \frac{1}{1+\frac{2x^2}{11}}\,dx = \frac{7}{11}\int \frac{1}{1+\left(\frac{\sqrt{2}x}{\sqrt{11}}\right)^2}\,dx$$

Let $u = \frac{\sqrt{2}x}{\sqrt{11}}$, then $du = \frac{\sqrt{2}}{\sqrt{11}}dx$, or yet $dx = \frac{\sqrt{11}}{\sqrt{2}}du$. Then:

$$\frac{7}{11}\int \frac{1}{1+\left(\frac{\sqrt{2}x}{\sqrt{11}}\right)^2}\,dx = \frac{7}{11}\cdot\frac{\sqrt{11}}{\sqrt{2}}\int \frac{1}{1+u^2}\,du = \frac{7}{\sqrt{11}\cdot\sqrt{2}}\arctan u + C$$

$$= \frac{7}{\sqrt{22}}\arctan\left(\sqrt{\frac{2}{11}}\,x\right) + C$$

Question 3b

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$. Let $u = \cos x$, so $du = -\sin x \, dx$,

$\int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C =$

$= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$

Question 3c

$\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx$. Let $u = \ln x$. Then $du = \frac{1}{x} dx$, so:

$$\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx = \int \frac{1}{\sqrt{1-\underbrace{(\ln x)^2}_{=u^2}}} \underbrace{\frac{1}{x} dx}_{=du} = \int \frac{1}{\sqrt{1-u^2}} du$$

$= \arcsin u + C = \arcsin(\ln x) + C$.

Question 4

Let $A(t) = A_0 e^{kt}$, with t measured in trillions of a ~~second~~ second. Then $A(3) = \frac{A_0}{2}$, so

$A(3) = \frac{A_0}{2} = A_0 e^{k3} \Rightarrow \frac{1}{2} = e^{3k} \Rightarrow 3k = \ln\left(\frac{1}{2}\right)$

$\Rightarrow k = \frac{1}{3}\ln\left(\frac{1}{2}\right)$. So $A(t) = A_0 e^{kt} = A_0 e^{\frac{t}{3}\ln\left(\frac{1}{2}\right)} =$

$= A_0 e^{\ln\left(\frac{1}{2}\right)^{t/3}}$. But $e^{\ln\left(\frac{1}{2}\right)^{t/3}} = \left(\frac{1}{2}\right)^{t/3}$ by the property $e^{\ln x} = x$.

So $A(t) = A_0 \left(\frac{1}{2}\right)^{t/3}$.

Question 5

$$\frac{dP}{dt} \propto P^2 + B \quad \text{or} \quad \frac{dP}{dt} = k(P^2+B) \Rightarrow \int \frac{dP}{B+P^2} = k\int dt$$

$\Rightarrow \frac{1}{\sqrt{B}} \arctan\left(\frac{P}{\sqrt{B}}\right) = kt + C_1$. Let P_0 be the initial population, so $P(0) = P_0$ and $\frac{1}{\sqrt{B}} \arctan\left(\frac{P_0}{\sqrt{B}}\right) = k \cdot 0 + C_1 \Rightarrow C_1 = \frac{1}{\sqrt{B}} \arctan\left(\frac{P_0}{\sqrt{B}}\right)$.

Then $\frac{1}{\sqrt{B}} \arctan\left(\frac{P}{\sqrt{B}}\right) = kt + \frac{1}{\sqrt{B}} \arctan\left(\frac{P_0}{\sqrt{B}}\right)$. After 3 weeks the population is $2P_0$, so:

$$\frac{1}{\sqrt{B}} \arctan\left(\frac{2P_0}{\sqrt{B}}\right) = k \cdot 3 + \frac{1}{\sqrt{B}} \arctan\left(\frac{P_0}{\sqrt{B}}\right)$$

$\Rightarrow k = \frac{1}{3\sqrt{B}} \left[\arctan\left(\frac{2P_0}{\sqrt{B}}\right) - \arctan\left(\frac{P_0}{\sqrt{B}}\right)\right]$. Hence

$$\arctan\left(\frac{P}{\sqrt{B}}\right) = \frac{t}{3}\left[\arctan\left(\frac{2P_0}{\sqrt{B}}\right) - \arctan\left(\frac{P_0}{\sqrt{B}}\right)\right] + \arctan\left(\frac{P_0}{\sqrt{B}}\right)$$

or yet:

$$P(t) = \sqrt{B} \tan\left(\frac{t}{3}\left[\arctan\left(\frac{2P_0}{\sqrt{B}}\right) - \arctan\left(\frac{P_0}{\sqrt{B}}\right)\right] + \arctan\left(\frac{P_0}{\sqrt{B}}\right)\right)$$

Question 6 Newton's law of cooling

$$\frac{dT}{dt} = k(T - T_s) \Rightarrow T = Ae^{kt} + T_s.$$

Consider 1:30pm as time zero. Then, using $T_s = 20$

$$T(0) = 32.5 = Ae^{k\cdot 0} + 20 \Rightarrow A = 12.5$$

$T = 12.5e^{kt} + 20$. After one hour, $T = 30.3$, so

$$T(1) = 30.3 = 12.5e^{k\cdot 1} + 20 \Rightarrow e^k = \frac{30.3 - 20}{12.5}$$

$$\Rightarrow k = \ln\left(\frac{10.3}{12.5}\right). \text{ Then } T = 12.5e^{t\ln\left(\frac{10.3}{12.5}\right)} + 20 =$$

$$= 12.5e^{\ln\left(\frac{10.3}{12.5}\right)^t} + 20 \Rightarrow T = 12.5\left(\frac{10.3}{12.5}\right)^t + 20$$

At the time of the murder, $T = 37$ so

$$37 = 12.5\left(\frac{10.3}{12.5}\right)^t + 20 \Rightarrow \frac{17}{12.5} = \left(\frac{10.3}{12.5}\right)^t$$

$$\Rightarrow \ln\left(\frac{17}{12.5}\right) = t\ln\left(\frac{10.3}{12.5}\right) \Rightarrow t = \ln\left(\frac{17}{12.5}\right) \Big/ \ln\left(\frac{10.3}{12.5}\right)$$

Notice that this value of t is negative. This is because we chose 1:30pm as the time zero. Hence the crime happened $\left|\dfrac{\ln\left(17/12.5\right)}{\ln\left(\frac{10.3}{12.5}\right)}\right|$ hours before 1:30pm.

Question 7a $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\tan 5x} = \dfrac{0}{0} \overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{(\sin 4x)'}{(\tan 5x)'} = \lim\limits_{x\to 0} \dfrac{4\cos 4x}{5\sec^2 5x} = \dfrac{4}{5}$

7b $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} = \dfrac{\infty}{\infty} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{(\ln x)'}{(\sqrt{x})'} = \lim\limits_{x\to\infty} \dfrac{1/x}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{1}{x}\cdot 2\sqrt{x}$

$= \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

7c $\lim\limits_{\theta\to\frac{\pi}{2}} \dfrac{1-\sin\theta}{\csc\theta} = 0$ since $\dfrac{1-\sin\theta}{\csc\theta} = \dfrac{1-\sin\theta}{\frac{1}{\sin\theta}} = (1-\sin\theta)\cdot\sin\theta$

No L'Hospital rule is necessary here.

7d $\lim\limits_{x\to 0} \dfrac{\cos mx - \cos nx}{x^2} = \dfrac{0}{0} \overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{-m\sin mx + n\sin nx}{2x} = \dfrac{0}{0}$

$\overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{-m^2\cos mx + n^2\cos nx}{2} = \dfrac{n^2-m^2}{2}$

7e $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \infty\cdot 0$. Write $\lim\limits_{x\to\infty} x^3 e^{-x^2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} = \dfrac{\infty}{\infty}$

$\overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \dfrac{3}{2}\lim\limits_{x\to\infty} \dfrac{x}{e^{x^2}} = \dfrac{\infty}{\infty} \overset{L'H}{=} \dfrac{3}{2}\lim\limits_{x\to\infty} \dfrac{1}{2xe^{x^2}} = 0$

7f $\lim\limits_{x\to\infty} x\tan\left(\frac{1}{x}\right) = \infty\cdot 0$, write $\lim\limits_{x\to\infty} \dfrac{\tan(\frac{1}{x})}{\frac{1}{x}} = \dfrac{0}{0} \overset{L'H}{=}$

$= \lim\limits_{x\to\infty} \dfrac{\sec^2\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \sec^2\frac{1}{x} = 1$

Question 7g $\lim_{t\to\infty} t^{\frac{\ln 2}{1+\ln t}} = \infty^0$. Let $y = t^{\frac{\ln 2}{1+\ln t}}$

Then $\ln y = \ln t^{\frac{\ln 2}{1+\ln t}} = \frac{\ln 2}{1+\ln t} \cdot \ln t = \frac{\ln 2}{\frac{1+\ln t}{\ln t}} = \frac{\ln 2}{\frac{1}{\ln t} + 1}$

when $t \to \infty$, $\frac{1}{\ln t} \to 0$, so

$\lim_{t\to\infty} \ln y = \lim_{t\to\infty} \frac{\ln 2}{\frac{1}{\ln t} + 1} = \ln 2$. Then $\lim_{t\to\infty} y = \lim_{t\to\infty} e^{\ln y} = e^{\ln 2} = 2$

7h $\lim_{x\to 1} (2-x)^{\tan(\frac{\pi x}{2})}$. $2-x \to 1$ when $x \to 1$ and

$\tan(\frac{\pi x}{2}) \to \tan\frac{\pi}{2} = \infty$, so this is of the form 1^∞.

Let $y = (2-x)^{\tan\frac{\pi x}{2}} \Rightarrow \ln y = \tan\frac{\pi x}{2} \ln(2-x) = \frac{\ln(2-x)}{\cot\frac{\pi x}{2}}$

$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln(2-x)}{\cot\frac{\pi x}{2}} = \frac{0}{0} \stackrel{L'H}{=} \lim_{x\to 1} \frac{\frac{-1}{2-x}}{-\frac{\pi}{2}\csc^2(\frac{\pi x}{2})} = \frac{2}{\pi}$

So $\lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^{\frac{2}{\pi}}$