MATH 155A FALL 13
PRACTICE MIDTERM 3, SOLUTIONS.

Question 1. Find all the maxima and minima of the functions on the given intervals.

(a) $f(x) = \cos^2 x - 2\sin x$, on $[0, 2\pi]$.

(b) $f(x) = x\sqrt{6-x}$, on $[-10, 6]$.

(c) $f(x) = 5x^{\frac{2}{3}} - 2x^{\frac{5}{3}}$, on $(-\infty, \infty)$.

Solution. (a) absolute minimum at $\frac{\pi}{2}$, local maximum at 0 and 2π, absolute maximum at $\frac{3\pi}{2}$. (b) absolute maximum at 4, absolute minimum at -10, local minimum at 6. (c) local maximum at 1 and local minimum at 0 (notice that f' is not defined at zero), no absolute maximum or minimum.

Question 2. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

Solution. Put $f(x) = 2x - 1 - \sin x$. One readily checks using the intermediate value theorem that there exists c such that $f(c) = 0$. To see that c is unique, suppose there is a d such that $f(d) = 0$, $d \neq c$. Then there must exist a $r \in (c, d)$ such that $f'(r) = 0$. But $f'(x) = 2 - \cos x$ is never zero since $\cos x \leq 1$.

Question 3. Find the limit or show that it does not exist.

(a) $\displaystyle\lim_{x\to\infty} \frac{x \sin x}{x^2+1}$.

(b) $\displaystyle\lim_{x\to-\infty} \left(x + \sqrt{x^2+2x}\right)$.

(c) $\displaystyle\lim_{x\to\infty} \frac{x^2 - x^4}{3x}$.

(d) $\displaystyle\lim_{x\to-\infty} \frac{2x^5 + x^4 - 3x^2 + 7}{x(2x - 4x^3 - 5x^4)}$.

(e) $\displaystyle\lim_{x\to\infty} \sqrt{x}\sin\frac{1}{x}$.

Solutions. (a) 0 (squeeze theorem). (b) -1 (multiply and divide by $x - \sqrt{x^2+2x}$). (c) $-\infty$ (factor x^4 on the top and x on the bottom). (d) $-\frac{2}{5}$ (factor x^5 on the top and on the bottom). (e) 0 (write $\sqrt{x} = \frac{x}{\sqrt{x}}$ and use $\frac{\sin(1/x)}{1/x} \to 1$).

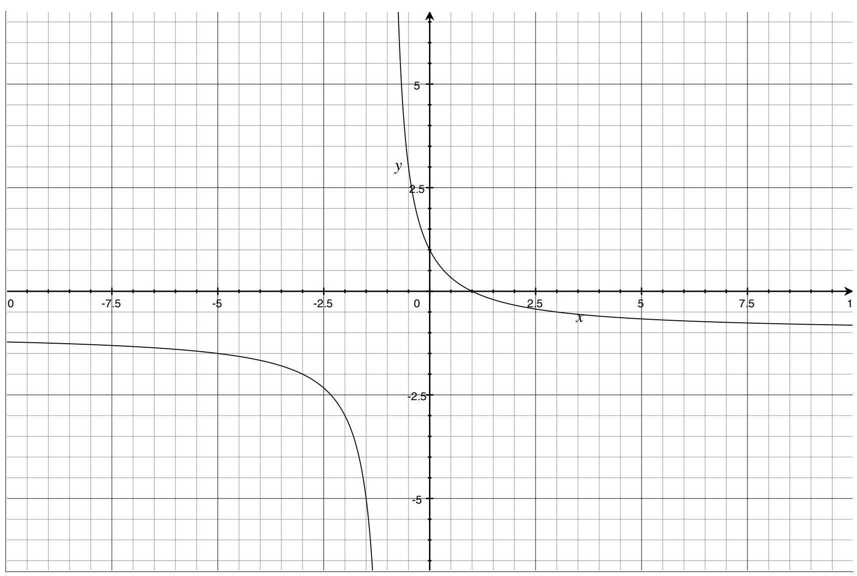
Question 4. Sketch the graph of the given functions.

(a) $y = \dfrac{1-x}{1+x}$.

(b) $y = \sqrt{x^2+x} - x$.

(c) $y = \dfrac{\sin x}{2+\cos x}$.

Solution.
(a) No critical points, H.A. $y = -1$, V.A. $x = -1$.



(b) Notice that the function is not defined for $-1 < x < 0$. Compute

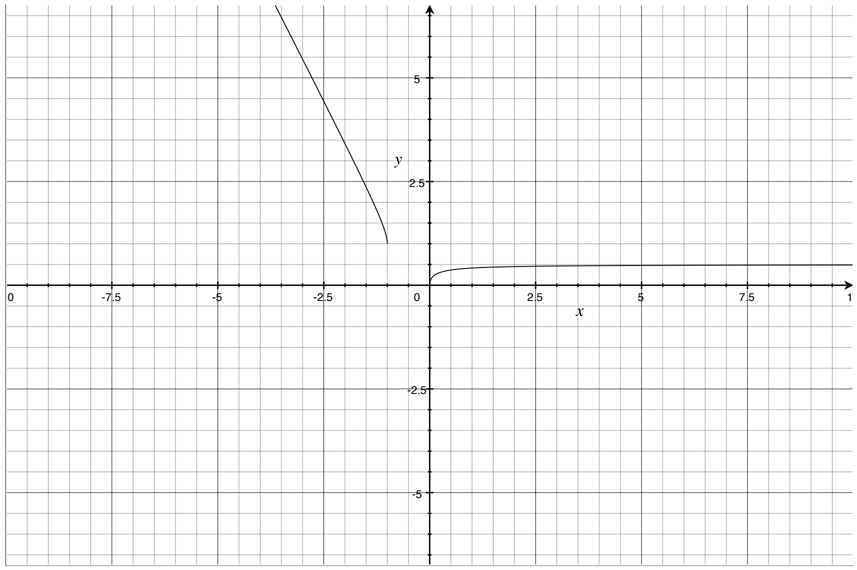
$$y' = \frac{2x+1}{2\sqrt{x^2+x}} - 1.$$

Notice that y' is never zero. In fact

$$y' = 0 \Leftrightarrow 2x+1 = 2\sqrt{x^2+x},$$

what gives, upon squaring, $4x^2+4x+1 = 4x^2+4x$ which has no solution.

Next, notice that $f'(x) < 0$ for $x < -1$ and $f'(x) > 0$ for $x \geq 0$ (f' is not defined at $x = -1$). We also check that $y'' < 0$ for all points in the domain. Finally, $y = \frac{1}{2}$ is a H.A. and there are no V. A.



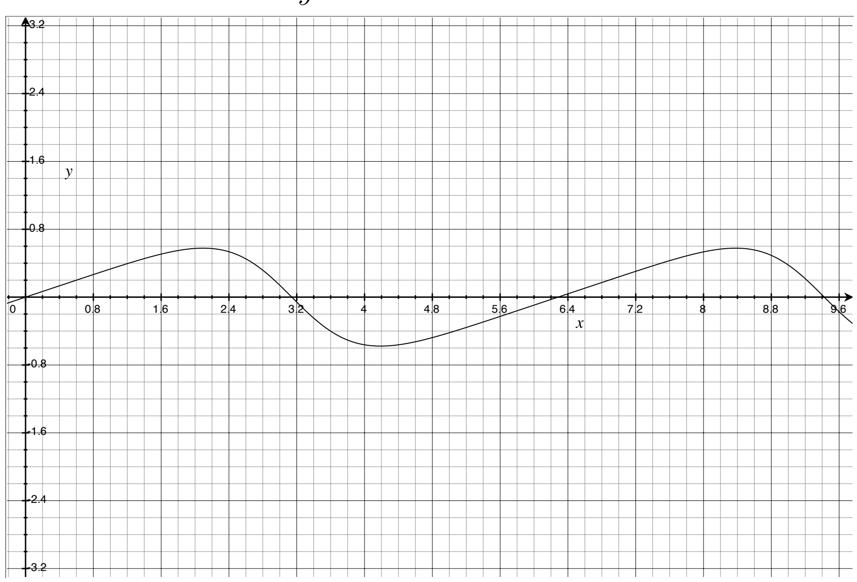
(c) y is periodic with period 2π, so it suffices to graph it on $[0, 2\pi]$. Compute

$$y' = \frac{2\cos x + 1}{(2+\cos x)^2},$$

and

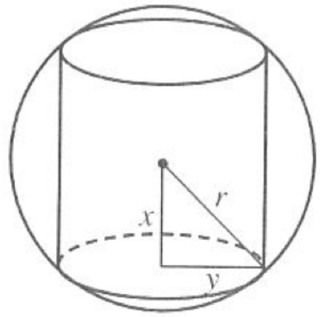
$$y'' = \frac{-2\sin x(1-\cos x)}{(2+\cos x)^3}.$$

It follows that there is a maximum for $x = \frac{2\pi}{3}$, a minimum for $x = \frac{4\pi}{3}$, and an inflection point for $x = \pi$. $y'' < 0$ for $0 < x < \pi$ and $y'' > 0$ for $\pi < x < 2\pi$.



Question 5. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such cylinder.

Solution. First we draw a picture:



The cylinder has surface area

$$2(\text{area of the base}) + (\text{lateral surface area}) = 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}).$$

Thus

$$S = 2\pi y^2 + 2\pi y(2x).$$

Using $x^2 + y^2 = r^2$, we find

$$S = 2\pi r^2 - 2\pi x^2 + 4\pi x\sqrt{r^2 - x^2},$$

for $0 \leq x \leq r$. Computing S' and following the standard optimization procedure, one finds

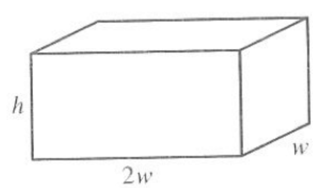
$$x = r\sqrt{\frac{5-\sqrt{5}}{10}},$$

from which we obtain that the maximum surface area is

$$\pi r^2(1+\sqrt{5}).$$

Question 6. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$ 10 per square meter. Material for the sides costs \$ 6 per squre meter. Find the cost of materials for the cheapest container.

Solution. First we draw a picture:

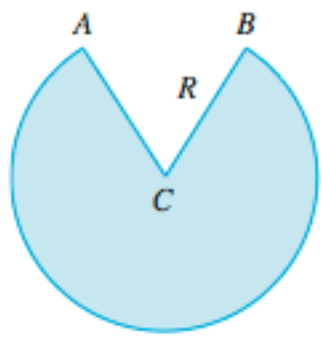


The volume is $V = lwh = 2w^2h = 10$ so that $h = \frac{5}{w^2}$. The cost is then

$$C = 20w^2 + 36wh = 20w^2 + \frac{180}{w}.$$

Following the standard optimization procedure we find $w = \sqrt[3]{\frac{9}{2}}$ and the cheapest cost to be \$ 163.54.

Question 7. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



Solution. The resulting cup has the form of a cone with height h, and a basis of radius r. Then $h^2 + r^2 = R^2$, so that the volume becomes

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(R^2 h - h^3).$$

We quickly check that $h = \frac{R}{\sqrt{3}}$ yields the absolute maximum, and the maximum volume is therefore $\frac{2\pi R^3}{9\sqrt{3}}$.

Question 8. Find f if

(a) $f''(x) = 6x + \sin x$.

(b) $f'(x) = x^{-\frac{1}{3}}$, $f(1) = 1$, $f(-1) = -1$.

(c) $f'''(x) = \cos x$, $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$.

Solution. (a) $f(x) = x^3 - \sin x + Cx + D$. (b) $f(x) = \frac{3}{2}x^{\frac{2}{3}} - \frac{1}{2}$ if $x > 0$, and $f(x) = \frac{3}{2}x^{\frac{2}{3}} - \frac{5}{2}$ if $x < 0$. (c) $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1$.

Question 9. Use the form of the definition of the integral as a limit of sums to evaluate the integral.

(a) $\displaystyle\int_1^4 (x^2 - 4x + 2)dx$.

(b) $\displaystyle\int_2^4 (x^3 - 1)dx$.

Solutions. (a) -3. (b) 58.

Question 10. Express the integral as a limit of Riemann sums. Do not evaluate the limit.

$$\int_0^{2\pi} x^2 \sin x \, dx.$$

Solution.

$$\int_0^{2\pi} x^2 \sin x \, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (i\frac{2\pi}{n})^2 \sin(i\frac{2\pi}{n})\frac{2\pi}{n}.$$

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`