MATH 155A FALL 13
MORE EXAMPLES WITH LIMITS

Evaluate the following limits, when possible.

(a)

$$\lim_{x\to\frac{\pi}{2}^-} \frac{(x-\frac{\pi}{2})^4}{\tan x}.$$

(b)

$$\lim_{\theta\to\pi} |\csc\theta|.$$

Solutions.

(a) We cannot plug in $x = \frac{\pi}{2}$ because $\tan\frac{\pi}{2}$ is undefined. Notice, however, that when x approaches $\frac{\pi}{2}$ from the left, the values of $\tan x$ become positive and unbounded (recall the graph of tan), i.e.

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty.$$

Therefore, $\frac{1}{\tan x}$ is bounded by, say, one, for values of x sufficiently close but less than $\frac{\pi}{2}$, and then

$$0 \le \frac{1}{\tan x} \le 1 \text{ for } x \text{ sufficiently close to } \frac{\pi}{2} \text{ and } x < \frac{\pi}{2}.$$

Multiplying by $(x-\frac{\pi}{2})^4$:

$$0 \le \frac{(x-\frac{\pi}{2})^4}{\tan x} \le (x-\frac{\pi}{2})^4 \text{ for } x \text{ sufficiently close to } \frac{\pi}{2} \text{ and } x < \frac{\pi}{2}.$$

Since

$$\lim_{x\to\frac{\pi}{2}^-} (x-\frac{\pi}{2})^4 = 0,$$

by the squeeze theorem we conclude that

$$\lim_{x\to\frac{\pi}{2}^-} \frac{(x-\frac{\pi}{2})^4}{\tan x} = 0.$$

(b) Recall that $\csc\theta = \frac{1}{\sin\theta}$. Since $\sin\pi = 0$, we cannot simply plug in $\theta = \pi$. Recall also that

$$|x| = \begin{cases} x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

For θ less than and close enough to π, we have $\sin\theta > 0$, whereas for θ greater than and close enough to π, we have $\sin\theta < 0$ (recall the graph of sin). Therefore,

$$\lim_{\theta\to\pi^-} \frac{1}{\sin\theta} = \infty,$$

since $\sin\theta$ approaches zero by positive values, and

$$\lim_{\theta\to\pi^+}\frac{1}{\sin\theta} = -\infty,$$

since $\sin\theta$ approaches zero by negative values. Thus,

$$\lim_{\theta\to\pi^-}|\csc\theta| = \lim_{\theta\to\pi^-}\left|\frac{1}{\sin\theta}\right| = \lim_{\theta\to\pi^-}\frac{1}{\sin\theta} = \infty,$$

and

$$\lim_{\theta\to\pi^+}|\csc\theta| = \lim_{\theta\to\pi^+}\left|\frac{1}{\sin\theta}\right| = \lim_{\theta\to\pi^-}\left(-\frac{1}{\sin\theta}\right) = \infty.$$

As both limits from the right and from the left agree, we obtain

$$\lim_{\theta\to\pi}|\csc\theta| = \infty.$$

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