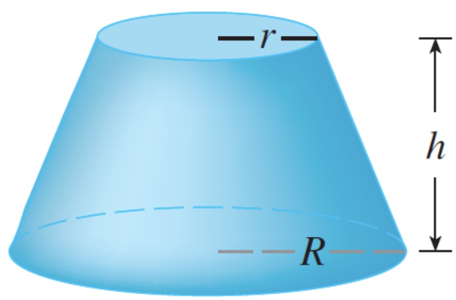
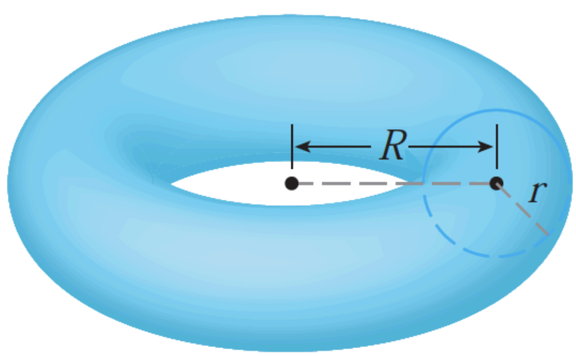
MATH 155A FALL 13
EXAMPLES SECTION 5.2.

Question 1. Find the volume of a frustum of a right circular cone with height h, lower base of radius R, and top of radius r, as shown in the picture:

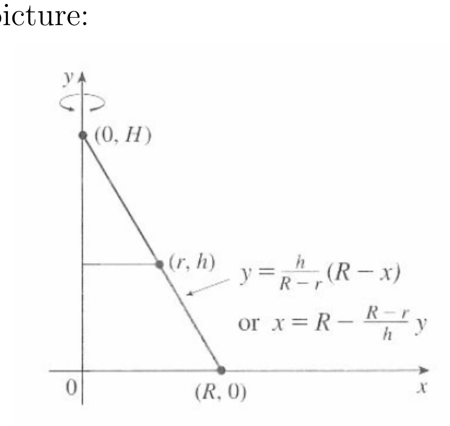


Question 2. Find the volume of a solid torus, the donut-shaped solid shown in the figure below, of radii r and R.



(pictures from Stewart's Calculus).

Solution 1. Consider the picture:



The solid is obtained by rotating the region with $0 \leq y \leq h$ about the y-axis. Cross-sections perpendicular to the y-axis are disks of radius x. Notice that

$$y = \frac{h}{R-r}(R-x),$$

or

$$x = R - \frac{R-r}{h}y.$$

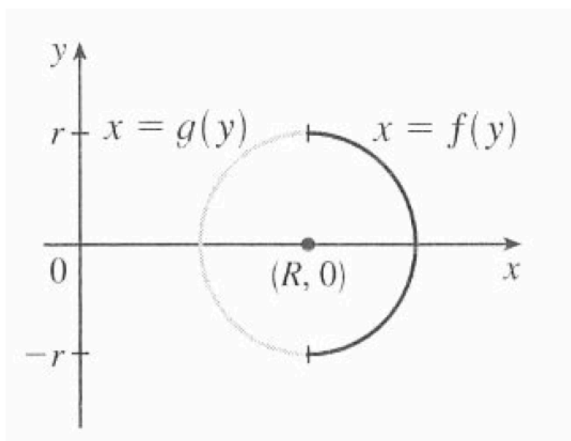
Then

$$A = \pi(\text{radius})^2 = \pi x^2 = \pi(R - \frac{R-r}{h}y)^2.$$

We find

$$\begin{aligned} V &= \int_0^h \pi(R - \frac{R-r}{h}y)^2\,dy \\ &= \frac{1}{3}\pi h(R^2 + Rr + r^2). \end{aligned}$$

Solution 2. Consider the picture:



The torus is obtained by rotating the circle $(x-R)^2 + y^2 = r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by

$$x = R + \sqrt{r^2 - y^2} = f(y),$$

and the left half by

$$x = R - \sqrt{r^2 - y^2} = g(y).$$

Notice that $f(y)$ is the outer radius and $g(y)$ the inner radius, thus

$$\begin{aligned} V &= \pi \int_{-r}^{r} \left((f(y))^2 - (g(y))^2\right) dy \\ &= 2\pi \int_0^r \left[\left(R^2 + 2R\sqrt{r^2-y^2} + r^2 - y^2\right) - \left(R^2 - 2R\sqrt{r^2-y^2} + r^2 - y^2\right)\right] dy \end{aligned}$$

$$= 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy.$$

We haven't yet learned how to compute this integral. However, interpreting it in terms of area, we notice that it represents a quarter of the area of a circle of radius r, thus

$$\begin{aligned} V &= 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy \\ &= 8\pi R \frac{\pi r^2}{4} \\ &= 2\pi^2 r^2 R. \end{aligned}$$

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`