MATH 155A FALL 13
EXAMPLES SECTION 4.1.

Question. Find the area under the graph of $y = x^2$ between $x = 1$ and $x = 3$.

Solutions.

Set

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n},$$
$$x_i = a + i\Delta x = 1 + \frac{2}{n}i.$$

We have, with $f(x) = x^2$,

$$f(x_i) = x_i^2 = (1 + \frac{2}{n}i)^2.$$

The Riemann sum becomes

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n}(1 + \frac{2}{n}i)^2 \frac{2}{n} \\ &= \frac{2}{n}\sum_{i=1}^{n}(1 + \frac{2}{n}i)^2 \\ &= \frac{2}{n}\sum_{i=1}^{n}(1 + \frac{4}{n}i + \frac{4}{n^2}i^2) \\ &= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2. \end{aligned}$$

Noticing that

$$\sum_{i=1}^{n} 1 = n,$$

and recalling

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

we find

$$R_n = 2 + \frac{8n(n+1)}{2n^2} + \frac{8n(n+1)(2n+1)}{6n^3}.$$

We finally find

$$\begin{aligned} A &= \lim_{n\to\infty} R_n \\ &= \lim_{n\to\infty} \left(2 + \frac{8n(n+1)}{2n^2} + \frac{8n(n+1)(2n+1)}{6n^3}\right) \\ &= 2 + \lim_{n\to\infty} \frac{8n(n+1)}{2n^2} + \lim_{n\to\infty} \frac{8n(n+1)(2n+1)}{6n^3} \\ &= 2 + 4 + \frac{16}{6} = \frac{26}{3}. \end{aligned}$$

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`