Stony Brook University.

MAT 127 — Calculus C, Spring 12.

How come $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges?

The goal of this sheet is to give further intuition on the divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

When first presented with the concept of an infinity series, one may (reasonably) think that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converges. After all, when n is very large, the fraction $\frac{1}{n}$ becomes very small, and hence we are adding very tiny numbers. However, as we have seen in class, the series diverges. The basic idea is that, *although the terms of the series are getting very small, they are not becoming small fast enough to compensate for the fact that there are infinitely many of them.*

In class, we showed the divergence by using a trick of grouping the terms of partial sums into finite blocks which add to more than $\frac{1}{2}$. Since in the limit there are infinitely many of such blocks, the infinite series diverges.

Although the formal proof given in class settles the matter, some people may want to see a more concrete calculation. We will do this here.

Consider the partial sum of the first N terms of the harmonic series, i.e.

$$S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \cdots + \frac{1}{N}$$

Notice that the series is then given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n}$$

Be careful to not mix the indices n and N.

One can now compute the value of the sum S_N for different values of N, say, $N = 10$, $N = 100$, $N = 1000$, etc. While this would be a tedious task even with a simple calculator at hand, it can be easily done with the help of the software Mathematica (the code is given at the end). We find:

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n} = 2.92897$$

$$S_{100} = \sum_{n=1}^{100} \frac{1}{n} = 5.18738$$

$$S_{1000} = \sum_{n=1}^{1000} \frac{1}{n} = 7.48547$$

$$S_{10\,000} = \sum_{n=1}^{10\,000} \frac{1}{n} = 9.78761$$

$$S_{100\,000} = \sum_{n=1}^{100\,000} \frac{1}{n} = 12.0901$$

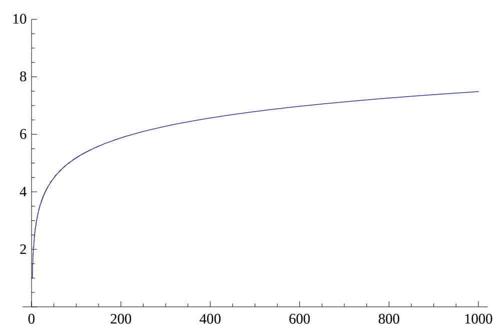
The reader can see that the more terms we add, the larger the sum — even if $\frac{1}{n}$ becomes very small when n is big.

Still not convinced that the series diverges? Let's compute

$$S_{1\,000\,000} = \sum_{n=1}^{1\,000\,000} \frac{1}{n} = 14.3927$$

The skeptical student is invited to continue this process: use larger and larger values of N, adding more and more terms, and see that the value of the sum continues to increase, without approaching any specif value.

It is also interesting to plot a graph of $\sum_{n=1}^{N} \frac{1}{n}$ against N. I.e., plot on the horizontal axis the number N of terms used in the sum and on the vertical axis the value of the sum. This is done in the graph below.



Finally, here is the Mathematica code:

```
f[N_] := Sum[1/n, {n, 1, N}]

f[10] // N
f[100] // N

2.92897

5.18738

f[1000] // N
f[10000] // N
f[100000] // N

7.48547

9.78761

12.0901

f[1000000] // N

14.3927

Plot[f[N], {N, 1, 1000}, PlotRange → {0, 10}]
```