Stony Brook University.
MAT 127 — Calculus C, Spring 12.
Examples for section 7.1

PROBLEMS

Question 1. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P(1 - \frac{P}{4200})$$

For what values of P is the population increasing/decreasing? What are the equilibrium solutions?

Question 2. For what values of r does the function $y = e^{rx}$ satisfy the differential equation

$$2y'' + y' - y = 0 \ ?$$

Question 3. Consider again the differential equation of question 2. If r_1 and r_2 are the values of r you found, show that the function $y = Ae^{r_1 x} + Be^{r_2 x}$ also satisfies the equation for any values of the constants A and B.

Question 4. Consider the simple harmonic oscillator equation:

$$x'' + \omega^2 x = 0$$

where $\omega = \sqrt{\frac{k}{m}}$ is the frequency of the system (so ω is simply a constant). Using power series, show that solutions are given by combinations of trigonometric functions.

Remark: Problem 4 is a more challenging one, and it should be read only after the basic material is well understood.

SOLUTIONS

1. Recall from calculus that a function is increasing if its derivative is positive and decreasing if its derivative is negative. The differential equation

$$\frac{dP}{dt} = 1.2P(1 - \frac{P}{4200})$$

gives a formula for the derivative. So $\frac{dP}{dt} > 0$ is equivalent to $1.2P(1 - \frac{P}{4200}) > 0$ and $\frac{dP}{dt} < 0$ is equivalent to $1.2P(1 - \frac{P}{4200}) < 0$. To solve these inequalities it is easier to first find the equilibrium solutions, i.e., those which satisfy $\frac{dP}{dt} = 0$ (these are called equilibrium because $\frac{dP}{dt} = 0$ implies that P is a constant, i.e., it doesn't change in time). We have

$$1.2P(1 - \frac{P}{4200}) = 0 \Rightarrow P = 0 \text{ or } P = 4200$$

Hence we have to find the sign of $1.2P(1 - \frac{P}{4200})$ for $P < 0$, $0 < P < 4200$, and $P > 4200$. For $P < 0$ one obtains $1.2P(1 - \frac{P}{4200}) < 0$, so $\frac{dP}{dt} < 0$ and P is decreasing there. For $0 < P < 4200$ we find $1.2P(1 - \frac{P}{4200}) > 0$, so $\frac{dP}{dt} > 0$ and therefore P is increasing on this interval. Finally, for $P > 4200$ we see that the population is decreasing.

Remark: Since P represents a population, which is never negative, we may ignore the values $P < 0$ in this problem.

2. Compute:

$$\begin{aligned} y &= e^{rx} \\ y' &= re^{rx} \\ y'' &= r^2e^{rx} \end{aligned}$$

Plugging in the equation one finds

$$(2r^2 + r - 1)e^{rx} = 0$$

Since the exponential function is never zero, the above equality implies that $2r^2 + r - 1 = 0$. Solving this quadratic equation for r we find $r_1 = \frac{1}{2}$ and $r_2 = -1$. Hence the functions $y_1 = e^{\frac{1}{2}x}$ and $y_2 = e^{-x}$ are solutions of the differential equation. In other words:

$$2y_1'' + y_1' - y_1 = 0 \tag{1}$$

$$2y_2'' + y_2' - y_2 = 0 \tag{2}$$

3. Put $y = Ae^{r_1x} + Be^{r_2x} = Ay_1 + By_2$. Then:

$$\begin{aligned} y' &= Ay_1' + By_2' \\ y'' &= Ay_1'' + By_2'' \end{aligned}$$

Plugging in the equation:

$$\begin{aligned} 2y'' + y' - y &= 2(Ay_1'' + By_2'') + Ay_1' + By_2' - (Ay_1 + By_2) \\ &= A(\underbrace{2y_1'' + y_1' - y_1}_{=0 \text{ by equation 1}}) + B(\overbrace{2y_1'' + y_1' - y_1}^{=0 \text{ by equation 2}}) \\ &= 0 \end{aligned}$$

So $2y'' + y' - y = 0$, as it had to be shown.

4. Let $x(t)$ be a solution of the differential equation. Consider its Maclaurin series:

$$x(t) = \sum_{n=0}^{\infty} c_n t^n \tag{3}$$

Compute

$$\begin{aligned} x'(t) &= \sum_{n=0}^{\infty} c_n n t^{n-1} = \sum_{n=1}^{\infty} c_n n t^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)t^n \\ x''(t) &= \sum_{n=0}^{\infty} c_{n+1}(n+1)nt^{n-1} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n \end{aligned}$$

Plugging in the equation we find

$$\begin{aligned} x'' + \omega^2 x &= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)t^n + \sum_{n=0}^{\infty} \omega^2 c_n t^n \\ &= \sum_{n=0}^{\infty} \Big(c_{n+2}(n+2)(n+1) + \omega^2 c_n \Big) t^n \end{aligned}$$

But since $x'' + \omega^2 x = 0$, we have that the above expression is zero. Recall that a polynomial is identically zero if and only if all its coefficients vanish. The same is true for power series. Hence

$$c_{n+2}(n+2)(n+1) + \omega^2 c_n = 0$$

leading to the following recursive formula

$$c_{n+2} = -\frac{\omega^2}{(n+2)(n+1)} c_n \tag{4}$$

We see that if we know c_0, then plugging $n = 0$ in (4) we obtain c_2 in terms of c_0; plugging $n = 2$ we get c_4 in terms of c_2, which in turn gives c_4 in terms of c_0, etc. In other words, all even coefficients can be expressed in terms of c_0:

$$\begin{aligned} c_2 &= -\frac{\omega^2}{2 \cdot 1} c_0 \\ c_4 &= -\frac{\omega^2}{4 \cdot 3} c_2 = \frac{\omega^4}{4 \cdot 3 \cdot 2 \cdot 1} c_0 \end{aligned}$$

Continuing we find

$$c_{2n} = \frac{(-1)^n \omega^{2n}}{(2n)!} c_0$$

Analogously all odd coefficients can be expressed in terms of c_1. One finds

$$\begin{aligned} c_3 &= -\frac{\omega^2}{3 \cdot 2} c_1 \\ c_5 &= -\frac{\omega^2}{5 \cdot 4} c_3 = \frac{\omega^4}{5 \cdot 4 \cdot 3 \cdot 2} c_1 \end{aligned}$$

And

$$c_{2n+1} = \frac{(-1)^n \omega^{2n}}{(2n+1)!} c_1$$

Plugging back into (3) and separating even and odd powers gives

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_{2n} t^{2n} + \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n} c_0}{(2n)!} t^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n} c_1}{(2n+1)!} t^{2n+1} \\ &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} t^{2n} + \frac{c_1}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} t^{2n+1} \end{aligned}$$

We recognize the first sum as the series for $\cos(\omega t)$ and the second one as the series for $\sin(\omega t)$, hence

$$x(t) = c_0 \cos(\omega t) + \frac{c_1}{\omega} \sin(\omega t)$$

The constants c_0 and c_1 are undetermined constants, that can be only found if we are given the initial conditions for this problem.