PROBLEMS

Question 1. A population is modeled by the differential equation
\[ \frac{dP}{dt} = 1.2P(1 - \frac{P}{4200}) \]
For what values of P is the population increasing/decreasing? What are the equilibrium solutions?

Question 2. For what values of r does the function \( y = e^{rx} \) satisfy the differential equation
\[ 2y'' + y' - y = 0 \]

Question 3. Consider again the differential equation of question 2. If \( r_1 \) and \( r_2 \) are the values of r you found, show that the function \( y = Ae^{r_1x} + Be^{r_2x} \) also satisfies the equation for any values of the constants A and B.

Question 4. Consider the simple harmonic oscillator equation:
\[ x'' + \omega^2 x = 0 \]
where \( \omega = \sqrt{\frac{k}{m}} \) is the frequency of the system (so \( \omega \) is simply a constant). Using power series, show that solutions are given by combinations of trigonometric functions.

Remark: Problem 4 is a more challenging one, and it should be read only after the basic material is well understood.

SOLUTIONS

1. Recall from calculus that a function is increasing if its derivative is positive and decreasing if its derivative is negative. The differential equation
\[ \frac{dP}{dt} = 1.2P(1 - \frac{P}{4200}) \]
gives a formula for the derivative. So \( \frac{dP}{dt} > 0 \) is equivalent to \( 1.2P(1 - \frac{P}{4200}) > 0 \) and \( \frac{dP}{dt} < 0 \) is equivalent to \( 1.2P(1 - \frac{P}{4200}) < 0 \). To solve these inequalities it is easier to first find the equilibrium solutions, i.e., those which satisfy \( \frac{dP}{dt} = 0 \) (these are called equilibrium because \( \frac{dP}{dt} = 0 \) implies that P is a constant, i.e., it doesn’t change in time). We have
\[ 1.2P(1 - \frac{P}{4200}) = 0 \Rightarrow P = 0 \text{ or } P = 4200 \]
Hence we have to find the sign of \( 1.2P(1 - \frac{P}{4200}) \) for \( P < 0 \), \( 0 < P < 4200 \), and \( P > 4200 \). For \( P < 0 \) one obtains \( 1.2P(1 - \frac{P}{4200}) < 0 \), so \( \frac{dP}{dt} < 0 \) and \( P \) is decreasing there. For \( 0 < P < 4200 \) we find \( 1.2P(1 - \frac{P}{4200}) > 0 \), so \( \frac{dP}{dt} > 0 \) and therefore \( P \) is increasing on this interval. Finally, for \( P > 4200 \) we see that the population is decreasing.
Remark: Since $P$ represents a population, which is never negative, we may ignore the values $P < 0$ in this problem.

2. Compute:

\begin{align*}
y &= e^{rx} \\
y' &= re^{rx} \\
y'' &= r^2 e^{rx}
\end{align*}

Plugging in the equation one finds

$$(2r^2 + r - 1)e^{rx} = 0$$

Since the exponential function is never zero, the above equality implies that $2r^2 + r - 1 = 0$. Solving this quadratic equation for $r$ we find $r_1 = \frac{1}{2}$ and $r_2 = -1$. Hence the functions $y_1 = e^{\frac{1}{2}x}$ and $y_2 = e^{-x}$ are solutions of the differential equation. In other words:

\begin{align*}
(1) & \quad 2y_1'' + y_1' - y_1 = 0 \\
(2) & \quad 2y_2'' + y_2' - y_2 = 0
\end{align*}

3. Put $y = Ae^{r_1x} + Be^{r_2x} = Ay_1 + By_2$. Then:

\begin{align*}
y' &= Ay_1' + By_2' \\
y'' &= Ay_1'' + By_2''
\end{align*}

Plugging in the equation:

$$2y'' + y' - y = 2(Ay_1'' + By_2'') + Ay_1' + By_2' - (Ay_1 + By_2)$$

$$= A(2y_1'' + y_1' - y_1) + B(2y_2'' + y_2' - y_2)$$

$$= A \underbrace{(2y_1'' + y_1' - y_1)}_{=0 \text{ by equation 1}} + B \underbrace{(2y_2'' + y_2' - y_2)}_{=0 \text{ by equation 2}}$$

$$= 0$$

So $2y'' + y' - y = 0$, as it had to be shown.

4. Let $x(t)$ be a solution of the differential equation. Consider its Maclaurin series:

\begin{equation}
x(t) = \sum_{n=0}^{\infty} c_n t^n
\end{equation}

Compute

\begin{align*}
x'(t) &= \sum_{n=0}^{\infty} c_n nt^{n-1} = \sum_{n=1}^{\infty} c_n nt^{n-1} = \sum_{n=0}^{\infty} c_{n+1} (n+1) t^n \\
x''(t) &= \sum_{n=0}^{\infty} c_{n+1} (n+1) nt^{n-1} = \sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1) t^n
\end{align*}
Plugging in the equation we find

\[ x'' + \omega^2 x = \sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1)t^n + \sum_{n=0}^{\infty} \omega^2 c_n t^n \]

\[ = \sum_{n=0}^{\infty} \left( c_{n+2} (n+2)(n+1) + \omega^2 c_n \right) t^n \]

But since \( x'' + \omega^2 x = 0 \), we have that the above expression is zero. Recall that a polynomial is identically zero if and only if all its coefficients vanish. The same is true for power series. Hence

\[ c_{n+2} (n+2)(n+1) + \omega^2 c_n = 0 \]

leading to the following recursive formula

(4)

\[ c_{n+2} = -\frac{\omega^2}{(n+2)(n+1)} c_n \]

We see that if we know \( c_0 \), then plugging \( n = 0 \) in (4) we obtain \( c_2 \) in terms of \( c_0 \); plugging \( n = 2 \) we get \( c_4 \) in terms of \( c_2 \), which in turn gives \( c_4 \) in terms of \( c_0 \), etc. In other words, all even coefficients can be expressed in terms of \( c_0 \):

\[ c_2 = -\frac{\omega^2}{2 \cdot 1} c_0 \]

\[ c_4 = -\frac{\omega^2}{4 \cdot 3} c_2 = -\frac{\omega^4}{4 \cdot 3 \cdot 2 \cdot 1} c_0 \]

Continuing we find

\[ c_{2n} = \frac{(-1)^n \omega^{2n}}{(2n)!} c_0 \]

Analogously all odd coefficients can be expressed in terms of \( c_1 \). One finds

\[ c_3 = -\frac{\omega^2}{3 \cdot 2} c_1 \]

\[ c_5 = -\frac{\omega^2}{5 \cdot 4} c_3 = -\frac{\omega^4}{5 \cdot 4 \cdot 3 \cdot 2} c_1 \]

And

\[ c_{2n+1} = \frac{(-1)^n \omega^{2n}}{(2n+1)!} c_1 \]

Plugging back into (3) and separating even and odd powers gives

\[ x(t) = \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_{2n} t^{2n} + \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n} c_0}{(2n)!} t^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n} c_1}{(2n+1)!} t^{2n+1} \]

\[ = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} t^{2n} + \frac{c_1}{\omega} \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} t^{2n+1} \]
We recognize the first sum as the series for $\cos(\omega t)$ and the second one as the series for $\sin(\omega t)$, hence

$$x(t) = c_0 \cos(\omega t) + \frac{c_1}{\omega} \sin(\omega t)$$

The constants $c_0$ and $c_1$ are undetermined constants, that can be only found if we are given the initial conditions for this problem.