HOMEWORK 9 - PART II

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

In this assignment, you will guided to construct solutions to the initial-value problem for the heat equation in $\mathbb{R}^n$:

$$u_t - \Delta u = 0 \text{ in } (0,\infty) \times \mathbb{R}^n. \tag{1}$$

Although this assignment has the complexity of a small class project, in that it is longer than a typical HW and stands as a self-contained topic, it will be graded as a regular homework.

Unless stated otherwise, the notation below is as in class.

Problem 1. Look for a solution to (1) in the form

$$u(t,x) = t^{-\alpha} v(t^{-\beta} x), \tag{2}$$

where α and β will be chosen and v will be determined. More precisely, proceed as follows:

(a) Show that plugging (2) into (1) produces

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0, \tag{3}$$

where $y := t^{-\beta} x$.

(b) Set $\beta = \frac{1}{2}$ in (3) to obtain

$$\Delta v(y) + \frac{1}{2} y \cdot \nabla v(y) + \alpha v(y) = 0. \tag{4}$$

(c) Assume that v is radially symmetric, i.e.,

$$v(y) = w(r), \tag{5}$$

where w is to be determined. Show that in this case (4) becomes

$$w'' + \frac{n-1}{r} w' + \frac{1}{2} r w' + \alpha w = 0. \tag{6}$$

(d) Set $\alpha = \frac{n}{2}$ in (6) to find

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0. \tag{7}$$

(e) From (7), conclude that

$$r^{n-1} w' + \frac{1}{2} r^n w = A, \tag{8}$$

where A is a constant.

(f) Set $A = 0$ in (8) and conclude that

$$w(r) = B e^{-\frac{1}{4} r^2}, \tag{9}$$

where B is a constant.

(g) Combine (2), (5), (9), and take into account the choices of α and β, to conclude that

$$u(t,x) = \frac{B}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ t > 0, \tag{10}$$

is a solution to (1).

The previous question motivates the following definition. The function

$$\Gamma(t,x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases}$$

is called the *fundamental solution of the heat equation.* Note that for $t > 0$, $\Gamma(t,x)$ is simply (10) with a specific choice of the constant B. This choice of B is to guarantee Γ to integrate to 1 (see the next question). In particular, $\Gamma(t,x)$ is a solution of (1).

Problem 2. Use the fact that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{\frac{n}{2}} \tag{11}$$

to show that for each $t > 0$

$$\int_{\mathbb{R}^n} \Gamma(t,x)\, dx = 1.$$

(You do *not* have to show (11).)

We now consider the initial-value problem for the heat equation:

$$u_t - \Delta u = 0, \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{12a}$$

$$u(0,x) = g(x),\ \ x \in \mathbb{R}^n. \tag{12b}$$

Define

$$u(t,x) := \int_{\mathbb{R}^n} \Gamma(t, x-y) g(y)\, dy,\ t > 0, x \in \mathbb{R}^n. \tag{13}$$

For the next questions, in (12), assume that $g \in C^0(\mathbb{R}^n)$ and that there exists a constant $C > 0$ such that $|g(x)| \le C$ for all $x \in \mathbb{R}^n$.

Problem 3. Show that (13) is well-defined.

Problem 4. Show that $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$, where u is defined by (13).
Hint: Use the following fact, that you do *not* need to prove. Let α be a multiindex and $t > 0$. If

$$\int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy$$

is well-defined, then

$$D^\alpha u(t,x) = \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy,$$

where we write D_x^α on the RHS to emphasize that the differentiation is with respect to the x variable.

Problem 5. Show that u given by (13) is a solution to the initial-value problem (12).
Hint: Use the following fact, that you do *not* need to prove. For each $x_0 \in \mathbb{R}^n$,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = g(x_0).$$

Problem 6. In (12), assume further that g has compact support and that $g \geq 0$. Show that for any $t > 0$ and any $x \in \mathbb{R}^n$, $u(t,x) \neq 0$. Explain why this can be interpreted as saying that, for the heat equation, information propagates at infinite speed. Contrast it with the finite speed of propagation for the wave equation.