HOMEWORK 4

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the following initial-boundary value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad \text{in } (0,\infty)\times(0,L),\\ u(t,0) = u(t,L) &= 0, \qquad t \geq 0,\\ u(0,x) &= g(x),\ \ 0 \leq x \leq L,\\ \partial_t u(0,x) &= h(x),\ \ 0 \leq x \leq L, \end{aligned}$$

where g and h are given and satisfy the compatibility conditions $g(0) = g(L) = 0 = h(0) = h(L)$. Use separation of variables to show that

$$u(t,x) = \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi c t}{L} + b_n \sin \frac{n\pi c t}{L} \right) \sin \frac{n\pi x}{L},$$

is a solution to the equation and satisfies the boundary conditions, where a_n and b_n are arbitrary coefficients.

Problem 2. Consider the following initial-boundary value problem for the heat equation in one dimension:

$$\begin{aligned} u_t - u_{xx} &= 0, \qquad \text{in } (0,\infty)\times(0,L),\\ u(t,0) = u(t,L) &= 0, \qquad t \geq 0,\\ u(0,x) &= g(x),\ \ 0 \leq x \leq L, \end{aligned}$$

where g is given and satisfies the compatibility conditions $g(0) = g(L) = 0$. Use separation of variables to show that

$$u(t,x) = \sum_{n=1}^{N} a_n e^{-\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L},$$

is a solution to the equation and satisfies the boundary conditions, where the a_n's are arbitrary coefficients. What happens when $t \to \infty$? Interpret your answer physically.

Problem 3. Find the Fourier series of the given functions:

(a) $f(x) = x$, $-1 \leq x \leq 1$.

(b) $f(x) = \sin(5x)$, $-\pi \leq x \leq \pi$.

Problem 4. Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0,\\ 0, & x = 0. \end{cases}$$

Show that $f \in C^0(\mathbb{R})$, that f is differentiable, but $f \notin C^1(\mathbb{R})$.

Problem 5.

(a) Prove that $C^k(I)$ is a vector space.

(b) Prove that the derivative $\frac{d}{dx}$ is a linear map between $C^k(I)$ and $C^{k-1}(I)$. What happens in the case $k = \infty$?