HOMEWORK 12

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the characteristics $(t, x(t, \alpha))$ for Burgers' equation $\partial_t u + u\partial_x u = 0$. Let v be u written with respect to (t, α) coordinates, i.e., $v(t, \alpha) := u(t, x(t, \alpha))$. Show that v satisfies $\partial_t v = 0$, so v is constant in time. Does this not contradict the fact that solutions to Burgers' equation with non-trivial compactly supported date blow up in finite time?

Problem 2. In this question, you will provide a blow-up proof for Burgers' equation different than the one given in class.

(a) Differentiate the equation and show that the variable $\psi := \partial_x u$ satisfies the equation

$$\partial_t \psi + u\partial_x \psi = -\psi^2. \tag{1}$$

(b) Show that (1) implies that $y(t) := \psi(t, x(t, \alpha))$ satisfies the Riccati equation $\dot{y} = -y^2$ along the characteristics $(t, x(t, \alpha))$.

(c) Use your knowledge of ODE to conclude, from the Riccati equation, blow-up for Burgers.

Problem 3. Define

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} := \sup_{x\in\mathbb{R}} |u(t, x)|.$$

Write Burgers' equation as an ODE along the characteristics (similarly to what you did for ψ in the previous problem) to conclude that

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|u(0, \cdot)\|_{L^\infty(\mathbb{R})} = \|h\|_{L^\infty(\mathbb{R})},$$

i.e., the L^∞ norm is conserved over time.