HOMEWORK 6 - PART II

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. For each set Ω below: (i) describe Ω in words (e.g., the first quadrant, intersection of the ball of radius one with the third quadrant, etc), drawing a picture when possible; (ii) identify $\partial\Omega$, and $\overline{\Omega}$; (iii) identify the area element of the boundary, i.e., dS; (iv) identify the normal to the boundary. The notation $B_r(z)$ is used for the (open) ball of radius r centered at $z \in \mathbb{R}^n$.

(a)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, |x| < 5 \right\}.$$

(b)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, -2 < x_1 < 2, -1 < x_2 < 1 \right\}.$$

(c)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1, -1 < x_3 < 1 \right\}.$$

(d)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1 \right\} \bigcap B_1(0).$$

(e)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}.$$

(f)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_3 > 0 \right\} \bigcap B_r(0).$$

(g)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, \langle x, (1,1,1) \rangle = 0 \right\} \bigcap B_1(0).$$

Problem 2. Recall the integration by parts formula in several dimensions:

$$\int_\Omega f \frac{\partial g}{\partial x_i} = -\int_\Omega \frac{\partial f}{\partial x_i} g + \int_{\partial\Omega} f g \nu_i. \tag{1}$$

Use (1) to prove the following formulas:

(a) Green's first identity:

$$\int_\Omega \nabla f \cdot \nabla g = -\int_\Omega f \Delta g + \int_{\partial\Omega} f \nabla g \cdot \nu.$$

(b) Green's second identity:

$$\int_\Omega (f\Delta g - g\Delta f) = \int_{\partial\Omega} (f\nabla g\cdot\nu - g\nabla f\cdot\nu).$$

(c)

$$\int_\Omega \Delta f = \int_{\partial\Omega} \nabla f\cdot\nu.$$

(d) Divergence theorem:

$$\int_\Omega \operatorname{div} F = \int_{\partial\Omega} F\cdot\nu.$$

In all questions below, let Ω be a domain in $\mathbb{R}^n$, i.e., Ω is an open and connected set contained in $\mathbb{R}^n$. For concreteness you can imagine that Ω is the ball of radius one centered at the origin.

Recall that we say that a function u is k-times continuously differentiable if all derivatives up to order k of u exist and are continuous.

Remember that we defined the spaces

$$C^k(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \text{ is } k\text{-times continuously differentiable } \Big\}.$$

Problem 3. Show that $C^k(\Omega)$ is a vector space.

Problem 4. Show that the Laplacian Δ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k\geq 2$.

Problem 5. Recall that

$$C^\infty(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \in C^k(\Omega) \text{ for every } k \Big\}.$$

Show that $C^\infty(\Omega)$ is a vector space and that the Laplacian Δ is a linear map from $C^\infty(\Omega)$ to itself.

Problem 6. Give a reasonable argument for why $C^k(\Omega)$ is an infinite-dimensional vector space. You are not asked to provide a mathematical and rigorous proof. Instead, you should use your knowledge of calculus and linear algebra, as well as the material we learned in class, to construct a sensible explanation, even if only an intuitive one (but if you do know a rigorous proof, that is welcome as well).

Problem 7. In class we said Ω has a C^k boundary if $\partial\Omega$ can be written locally as the graph of a C^k function. Make this definition more precise upon using mathematical quantifiers.

Problem 8. Show that if a_{ij} is symmetric in i and j, then ${a^i}_j = {a_i}^j$ so that we can write simply a^i_j, but that this is not the case otherwise.

Problem 9. Let a be a $n\times n$ matrix with entries ${a^i}_j$, where i the row and j the column. Show that the trace of a is given by ${a^i}_i$. If a is invertible with entries ${(a^{-1})^i}_j$, show that

$${a^i}_j {(a^{-1})^j}_\ell = \delta^i_\ell.$$

Show that the determinant of a is given by

$$\det(a) = \frac{1}{n!}\epsilon_{i_1 i_2\cdots i_n}\epsilon^{j_1 j_2\cdots j_n}{a^{i_1}}_{j_1}{a^{i_2}}_{j_2}\cdots {a^{i_n}}_{j_n}.$$

Above, $\epsilon_{i_1 i_2 \cdots i_n}$ is the n-dimensional totally anti-symmetric symbol, defined as $\epsilon_{i_1 i_2 \cdots i_n} = 1$ if $i_1, i_2, \cdots, i_n$ is an even permutation of $1, 2, \cdots, n$, $\epsilon_{i_1 i_2 \cdots i_n} = -1$ if $i_1, i_2, \cdots, i_n$ is an odd permutation of $1, 2, \cdots, n$, and $\epsilon_{i_1 i_2 \cdots i_n} = 0$ otherwise.
Hint: Show the determinant formula only for $n = 2$ and, perhaps, $n = 3$. The general case is too lengthy for you to spend time on this. You can, however, see the general proofs in textbooks if you are interested.