HOMEWORK 3

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

In the next two questions, you will be asked to perform some of the computations skipped in class in our derivation of solutions to the Schödinger equation for an electrostatic potential.

Question 1. Consider the ODE

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right)+\left(\lambda-\frac{m^2}{1-x^2}\right)\Phi=0. \tag{1}$$

Show that a solution to (1) is given by

$$\Phi(x)=(1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}P(x)}{dx^{|m|}}, \tag{2}$$

where P solves

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\lambda P=0. \tag{3}$$

Hint: Differentiate (3) a few times and observe the resulting pattern, or use induction, to conclude that if P solves (3) then

$$(1-x^2)\frac{d^{|m|+2}P}{dx^{|m|+2}}-2(|m|+1)x\frac{d^{|m|+1}P}{dx^{|m|+1}}+(\lambda-|m|(|m|+1))\frac{d^{|m|}P}{dx^{|m|}}=0. \tag{4}$$

Next, let $\widetilde{\Phi}$ be defined by

$$\Phi(x)=(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x), \tag{5}$$

where Φ is a solution to (1). Plugging (5) into (1), conclude that $\widetilde{\Phi}$ satisfies

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2}-2x\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+\left(\lambda-|m|(|m|+1)\right)\widetilde{\Phi}=0. \tag{6}$$

Compare (6) with (4) to obtain the result.

Question 2. Consider the ODE

$$\frac{1}{\varrho^2}\frac{d}{d\varrho}\left(\varrho^2\frac{dR}{d\varrho}\right)+\left(-\frac{1}{4}-\frac{\ell(\ell+1)}{\varrho^2}+\frac{\gamma}{\varrho}\right)R=0. \tag{7}$$

Show that (7) does not admit a non-trivial solution of the form

$$R(\varrho)=\sum_{k=0}^{\infty}a_k\varrho^k. \tag{8}$$

Hint: Plug (8) into (7) to derive

$$-\ell(\ell+1)a_0\varrho^{-2}+\left((2-\ell(\ell+1))a_1+\gamma a_0\right)\varrho^{-1}$$

$$+\sum_{k=0}^{\infty}\left(\left((k+3)(k+2)-\ell(\ell+1)\right)a_{k+2}+\gamma a_{k+1}-\frac{1}{4}a_k\right)\varrho^k=0. \tag{9}$$

From (9), conclude that $a_k = 0$ for all k.