HOMEWORK 7

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Prove the following fact that we used in the construction of solutions to Poisson's equation: let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, then

$$\lim_{r \to 0^+} \frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS = f(x).$$

Hint: Consider the difference $f(x) - \frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS$ and use $\frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS = 1$.

Problem 2. In class, we constructed solutions to Poisson's equation in $\mathbb{R}^n$ for $n \geq 3$. Carry out the construction in the case $n = 2$. You do *not* have to do all the steps. Rather, follow what was done in class and point out what changes in $n = 2$. This boils down to slightly modifying some of the estimates for the fundamental solution.

Problem 3. Let u be a non-trivial harmonic function in $\mathbb{R}^n$. Can u have compact support?
Hint: mean value theorem.

Problem 4. Prove the converse of the mean value theorem. I.e., let $u \in C^2(\Omega)$ be such that

$$u(x) = \frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \, dS$$

for each $B_r(x) \subset \Omega$. Show that $\Delta u = 0$ in Ω.
Hint: Assume that $\Delta u(x) \neq 0$ for some $x \in \Omega$. Use the functions $f(r)$, $f'(r)$ used in the proof of the mean value to derive a contradiction.

Problem 5. Give an interpretation of Poisson's equation in an application (e.g., in physics, biology, etc.).