HOMEWORK 8

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Prove the differentiation of moving regions formula stated in class:

$$\frac{d}{d\tau} \int_{\Omega(\tau)} f \, dx = \int_{\Omega(\tau)} \partial_\tau f \, dx + \int_{\partial\Omega(\tau)} f v \cdot \nu \, dS. \tag{1}$$

(See the class notes for the notation and precise assumptions.) For simplicity, prove (1) in the following particular case. Assume that $n = 3$ and that the domains $\Omega(\tau)$ are given by a one-parameter family of one-to-one and onto maps $\varphi = \varphi(\tau, x) : \Omega \to \Omega(\tau) = \varphi(\tau, \Omega)$, where $\Omega := \Omega(0)$ and $\varphi(0, \cdot) = \mathrm{id}_\Omega$, where id_Ω is the identity map on Ω, i.e., $\mathrm{id}_\Omega(x) = x$, $x \in \Omega$.

(a) For each fixed τ, consider the change of variables $x = \varphi(\tau, y)$, so that

$$\int_{\Omega(\tau)} f(\tau, x) \, dx = \int_\Omega f(\tau, \varphi(\tau, y)) J(\tau, y) \, dy, \tag{2}$$

where $J(\tau, y)$ is the Jacobian of the transformation $x = \varphi(\tau, y)$ for fixed τ.

(b) Show that there exists a on parameter family of vector fields $u(\tau, \cdot)$ such that

$$\partial_\tau \varphi(\tau, x) = u(\tau, \varphi(\tau, x)).$$

(c) Explain why $u = v$ on $\partial\Omega(\tau)$.

(d) Show that

$$\partial_\tau J(\tau, x) = (\operatorname{div} u)(\tau, \varphi(\tau, x)) J(\tau, x).$$

(e) Use (2) and the above to compute $\frac{d}{d\tau} \int_{\Omega(\tau)} f$, and do an integration by parts to obtain the result.

Problem 2. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^n$. Assume that u_0 and u_1 have their supports in the ball $B_R(0)$ for some $R > 0$. Show that $u = 0$ in the exterior of the region

$$I := \{ (t, x) \in (0, \infty) \times \mathbb{R}^n \mid x \in B_{R+t}(0) \}.$$

I is called a domain of influence for that data on $B_R(0)$ (compare with the $1d$ case).

Problem 3. Let u be a solution to the Cauchy problem for the wave equation and assume that u_0 and u_1 have compact support.

(a) Show that the energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left[(\partial_t u)^2 + |\nabla u|^2 \right] \, dx$$

is well-defined.

(b) Show that

$$E(t) = E(0),$$

i.e., the energy is conserved.

Problem 4. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^3$ with compactly supported data (i.e., u_0 and u_1 have compact support).

(a) Show that there exists a constant $C > 0$, depending on u_0 and u_1, such that

$$|u(t, x)| \leq \frac{C}{t}, \tag{3}$$

for $t \geq 1$ and $x \in \mathbb{R}^3$. Thus, for each fixed x, u approaches zero as $t \to \infty$, i.e., solutions decay in time.

Hint: Use the formula for solutions in $n = 3$. Since the data has compact support, it vanishes outside $B_R(0)$ for some $R > 0$. This implies an estimate for the area of the largest region within $B_t(x)$ where the data is non-trivial.

(b) Is the estimate (3) sharp? (I.e., can it be improved to show that solutions decay faster in time than $\frac{1}{t}$?)

(c) Do we still get decay if the data does not have compact support?