HOMEWORK 13

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the Cauchy problem for Burgers' equation with data given by

$$h(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

(a) Show that the solution is given by

$$u(t,x) = \begin{cases} 1, & x \le t, t < 1, \\ \frac{1-x}{1-t}, & t < x < 1, t < 1, \\ 0, & x \ge 1, t < 1. \end{cases}$$

(b) The denominator of $\frac{1-x}{1-t}$ approaches zero when $t \to 1^-$. Does that mean that $|u(t,x)| \to \infty$ as $t \to 1^-$? Does this not contradict your result from question 3 in HW 12? What exactly is becoming singular when the characteristics intersect at $(1,1)$?

(c) Let $0 < \beta < 1$ and define, for $t \ge 1$

$$\tilde{u}(t,x) = \begin{cases} 1, & x < \beta t + 1 - \beta, \\ 0 & x > \beta t + 1 - \beta. \end{cases}$$

Show that v given by

$$v(t,x) = \begin{cases} u(t,x), & 0 \le t < 1, \\ \tilde{u}(t,x), & t \ge 1, \end{cases}$$

is a weak solution if and only if $\beta = 1/2$. (This was essentially done in class. Here, you have to work out the calculations in more detail, including the case when Ω might intersect the region $\{t \le 1\}$.)

Problem 2. Show that the function v in the previous problem verifies the Rankine-Hugoniot conditions if an only if $\beta = 1/2$.

Problem 3. Prove that if u is a weak solution that is C^∞ then it is in fact a classical solution.

Problem 4. Formulate the definition of weak solutions, shocks, and the Rankine-Hugoniot conditions for systems of conservation laws. Give a brief sketch of the proof of the Rankine-Hugoniot theorem for systems (you do not have to do all the proof; it suffices to indicate how to modify the $N = 1$ case done in class. Your answer should be two or three sentences long.)

Problem 5. Show that the $1d$ compressible Euler equations form a system of conservation laws.