HOMEWORK 6 - PART I

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the following initial-value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \text{ in } (-\infty, \infty) \times (0, \infty), \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned} \tag{1}$$

(a) Solve (1) when $f(x) = x^2$ and $g(x) = 0$.

(b) Assume now that $c = 1$ and

$$f(x) = \begin{cases} 1, & -2 \leq x \leq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} -1, & -1 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Draw a diagram in the (x,t)-plane indicating the different regions where the solution is influenced by the initial condition f and g and the regions where the solution is identically zero. (You do **not** have to find u.) Is the solution to this problem a classical solution?

Problem 2. Consider the Cauchy problem for the $1d$ wave equation with data (u_0, u_1) and then with data $(\tilde{u}_0, \tilde{u}_1)$. Let u and $\tilde{u}$ be the corresponding solutions. Assume that on $[a,b]$ we have $u_0 = \tilde{u}_0$ and $u_1 = \tilde{u}_1$ Prove that $u = \tilde{u}$ in the domain of dependence with base $[a,b]$.

Problem 3. In the proof of existence of solutions to the initial-boundary value problem for the $1d$ wave equation (HW5, problem 1), drop the hypothesis $g''(0) = g''(L) = 0 = h''(0) = h''(L)$. What can you say about the formal solution in this case? (Will it be an actual solution in any sense? Classical, generalized?)

Problem 4. Write each PDE below in the form $F(x, u, Du, \ldots, D^m u) = 0$, i.e., identify the function F. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + u u_x = 0$.

(c) $a^{ijk} \partial^3_{ijk} v + v = 0$,

where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x+y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Problem 5. Consider a PDE $F(x, u, Du, \ldots, D^m u) = 0$. Prove that the PDE is linear (as defined in class in terms of linearity of F with respect to some of its entries) if and only if it can be written as

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha u = f,$$

as stated in class.