HOMEWORK 2

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider Euler's equations:

$$\partial_t \varrho + u^i \partial_i \varrho + \varrho \partial_i u^i = 0,$$
$$\varrho(\partial_t u^j + u^i \partial_i u^j) + \nabla^j p = 0,$$

where we recall that $p = p(\varrho)$. A fluid is called *incompressible* if $\varrho = constant$, in which case we can set $\varrho = 1$. In this case, the equations describing the fluid motion are

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$
$$\partial_i u^i = 0,$$

which are called the *incompressible Euler equations.* For an incompressible fluid, however, the pressure is no longer given by $p = p(\varrho)$, since the pressure would then be constant, but experiments show that the pressure can vary even if the density remains (approximately) constant. Show that in the case of the incompressible Euler equations, the pressure is given as a solution to

$$\Delta p = -\partial_j u^i \partial_i u^j.$$

Problem 2. Consider the incompressible Euler equations (see previous question):

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$
$$\partial_i u^i = 0.$$

The *vorticity* ω of the fluid is defined as

$$\omega := \operatorname{curl} u.$$

The vorticity is an important physical quantity; it measures, as the name suggests, "eddies" in the fluid. It is, therefore, important to know how it changes in time and space (i.e., what the dynamics of the vorticity is). Show that ω satisfies the following PDE:

$$\partial_t \omega + \nabla_u \omega - \nabla_\omega u = 0.$$

Above, the operators ∇_u and ∇_ω are defined as follows. For any vector field X, ∇_X is a short hand notation for $X \cdot \nabla$, i.e.,

$$\nabla_X := X \cdot \nabla,$$

where we recall that $X \cdot \nabla$ has been defined in class as

$$X \cdot \nabla = X^i \partial_i.$$

Problem 3. Show that in spherical coordinates the Laplacian reads

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2},$$

where

$$\Delta_{S^2} := \partial_\phi^2 + \frac{\cos\phi}{\sin\phi}\partial_\phi + \frac{1}{\sin^2\phi}\partial_\theta^2$$

is the Laplacian on the unit sphere, $r \in [0, \infty)$, $\phi \in [0, \pi]$, and $\theta \in [0, 2\pi)$.

Problem 4. Show that the Φ equation in the separation of variables for the Schrödinger equation can be written as

$$\frac{\sin\phi}{\Phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) - m^2 = -\lambda\sin^2\phi,$$

where $\lambda = \frac{2\mu}{\hbar^2}a$.