HOMEWORK 9 - PART I

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $1d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(t-s,y)\,dyds. \tag{1}$$

To do so, first use D'Alembert's formula to conclude that

$$u_s(t,x) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy.$$

Use the definition of u in terms of u_s and change variables to conclude (1).

Problem 2. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $3d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\,dy. \tag{2}$$

(The integrand in (2) is known as the retarded potential.) To do so, first use Kirchhoff's formula for solutions in $n=3$ to conclude that

$$u_s(t,x) = \frac{t-s}{\operatorname{vol}(\partial B_{t-s}(x))}\int_{\partial B_{t-s}(x)} f(s,y)\,dS(y).$$

Use the definition of u in terms of u_s and change variables to conclude (2).

Problem 3. Show that there exists a constant $C>0$ such that for any solution u to the $3d$ wave equation it holds that

$$|u(t,x)| \le \frac{C}{t}\int_{\mathbb{R}^3} \left(|D^2u_0(y)| + |Du_0(y)| + |u_0(y)| + |Du_1(y)| + |u_1(y)|\right) dy$$

for $t \ge 1$.

Hint: Use Kirchhoff's formula, note that for any function f we have

$$f(y) = f(y)\frac{y-x}{t}\cdot\frac{y-x}{t}$$

on $\partial B_t(x)$, and use one of Green's identities.

Problem 4. Consider continuous dependence on the data for the wave equation in $3d$, where smallness on the data part is measured with respect to the norm

$$\|f\|_2 := \int_{\mathbb{R}^3} \left(|D^2f(y)| + |Df(y)| + |f(y)|\right) dy.$$

Give a precise formulation of the continuous dependence on the data and prove your statement.

Hint: Use the estimate of problem 3 as the basis for your statement, and give a similar proof (now you have to also account for $t < 1$).