MATH 2300

Multivariable Calculus

Fall 2020

Table of contents

Derivatives and integrals of vector functions

Integrals

Arc lenght and curvature

Curvature

Normal and binormal vectors

Motion in space: velocity and acceleration

Functions of several variables

Level curves

Functions of three or more variables

Limits and continuity.

Continuity

Functions of three or more variables

Partial derivatives

Higher derivatives

Partial derivatives of functions of several variables

Tangent planes and linear approximations

Linear approximations

Differentiables

Functions of three or more variables

The chain rule

Directional derivatives and the gradient vector

Tangent planes and level surfaces

Maximum and minimal values

Absolute maximum and minimum values

Lagrange multipliers

Double integrals over rectangles

Iterated integrals

Double integrals over general regions

Surface area

Triple integrals

Volumes

Triple integrals in cylindrical coordinates

Triple integrals in spherical coordinates

The volume element in cylindrical and spherical coordinates

Change of variables in multiple integrals

- Triple integrals

Vector fields

Line integrals

The fundamental theorem of line integrals

Green's theorem

Curl and divergence

Parametric surfaces

- Tangent planes
- Surface area

Surface integrals

- Orientation
- Surface integrals of vector fields

Stokes' theorem

The divergence theorem

Abbreviations

LHS, RHS = left/right hand side

w.r.t. = with respect to

$\Rightarrow$ = implies

EX : example

Def = definition

Thm = theorem

Prop = proposition

$\square$ = end of proof

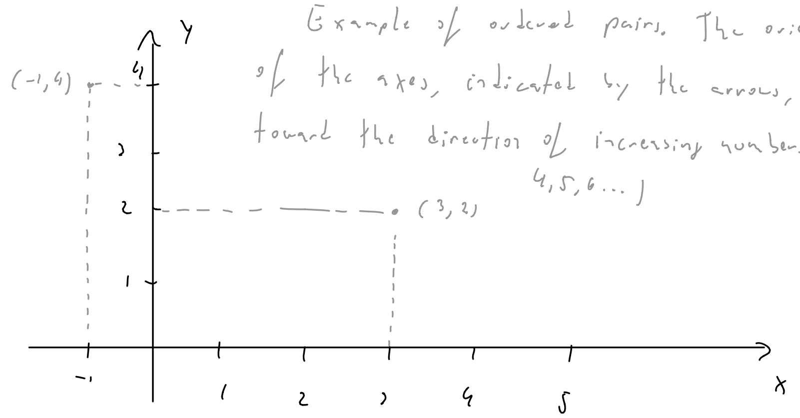
LHS := RHS means that the LHS is defined by the RHS

$\mathbb{R}$ = set of real numbers

nd = n dimensions, n dimensional, e.g., 1d, 2d, 3d.

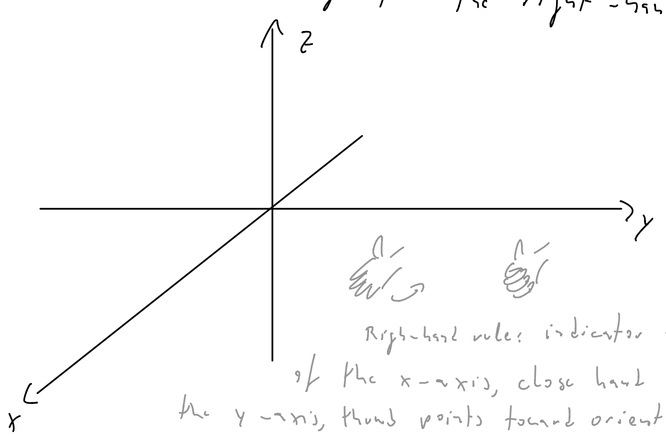
The three-dimensional coordinate system

From single variable calculus, recall that a point in the Cartesian plane can be represented by an <u>ordered pair</u> (a,b), where "ordered" means that the order in which the <u>coordinates</u> (i.e., the entries) a, b are presented matter, so $(a,b) \neq (b,a)$ (unless $a=b$).



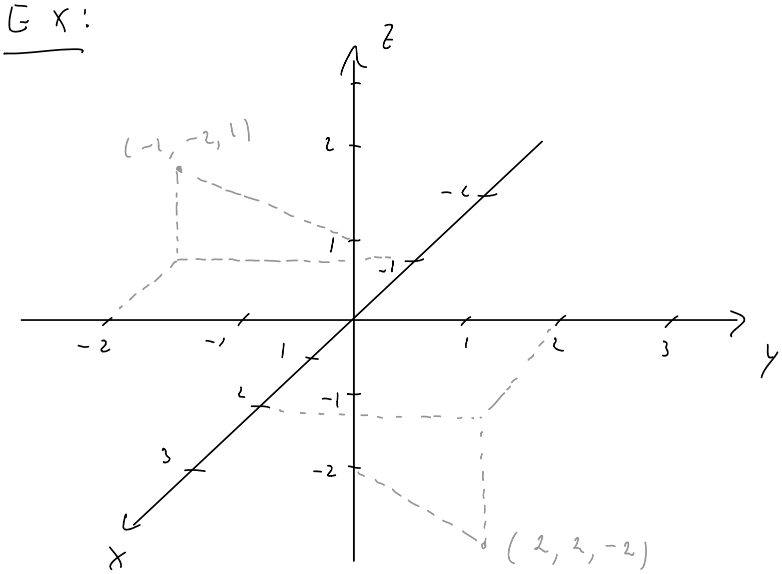
Example of ordered pairs. The orientation of the axes, indicated by the arrows, points toward the direction of increasing numbers (i.e., 4, 5, 6 ...)

Similarly, in three-dimensional space, we can represent points by <u>ordered triples</u> (a,b,c). We add to the xy-axes a third axis, the z-axis, perpendicular to the xy-plane, passing through the <u>origin</u> (i.e., where the axes meet), and oriented according to the right-hand rule:

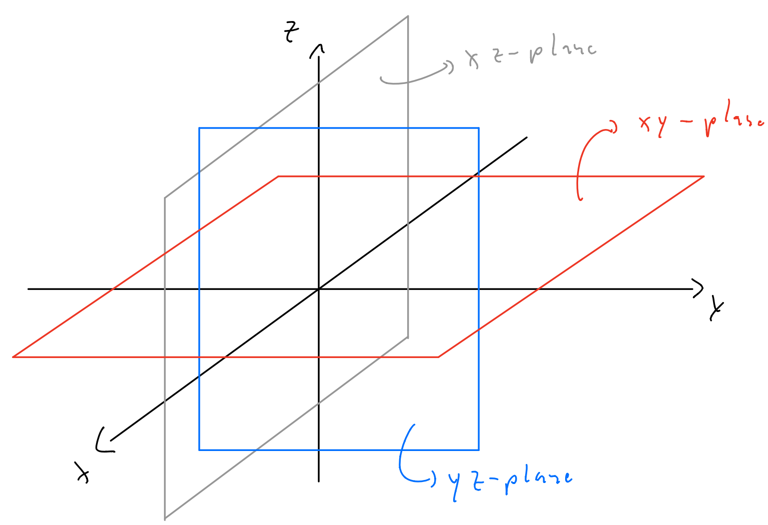


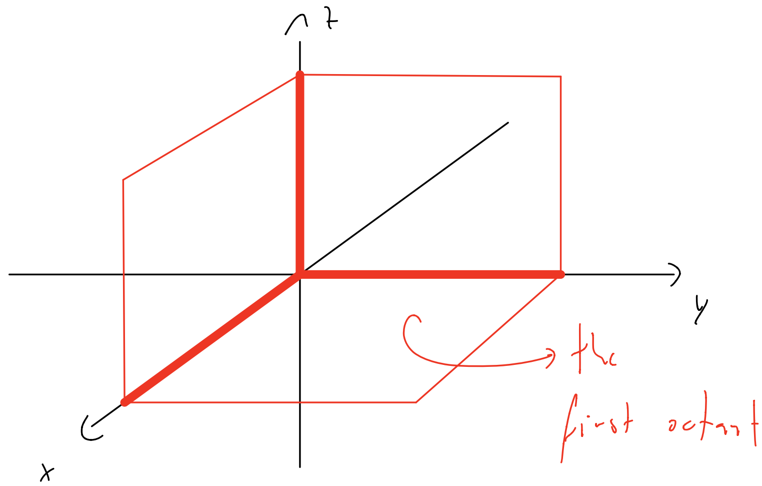
Right-hand rule: index toward the orientation of the x-axis, close hand toward orientation of the y-axis, thumb points toward orientation of the z-axis.

Ex:



If p is a point represented by (a, b, c), the numbers a, b, c are called the <u>coordinates</u> of the point p; the x-coordinate, y-coordinate, and z-coordinate, respectively. The xy-, xz-, and yz-plane are called the <u>coordinate planes</u> and they divide space into eight equal parts called <u>octants</u>.





We write $P(a,b,c)$ to indicate that P is a point with coordinates (a,b,c).

The Cartesian product

$$\mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (x,y,z) \mid x,y,z \in \mathbb{R} \}$$

is the set of all ordered triples and is called the three-dimensional rectangular coordinate system. It provides a a one-to-one correspondence between points in space and ordered triples in $\mathbb{R}^3$.

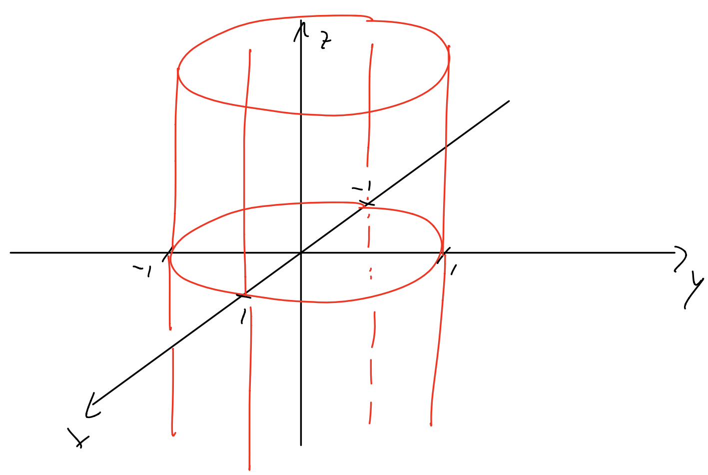
Surfaces

In 2d, an equation relating x and y defines a curve in the plane. In 3d, an equation relating x, y, and z defines a surface in $\mathbb{R}^3$.

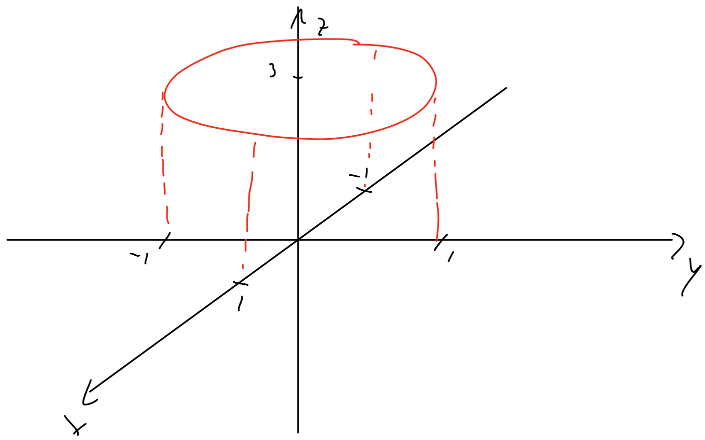
Ex: What surface is represented by the equations

$$x^2+y^2=1 \quad \text{and} \quad z=1 \quad ?$$

On the plane, $x^2+y^2=1$ is a circle of radius one centered at the origin. But in 3d, $x^2+y^2=1$ defines a cylinder of radius 1 and axis of symmetry equal to the z-axis:

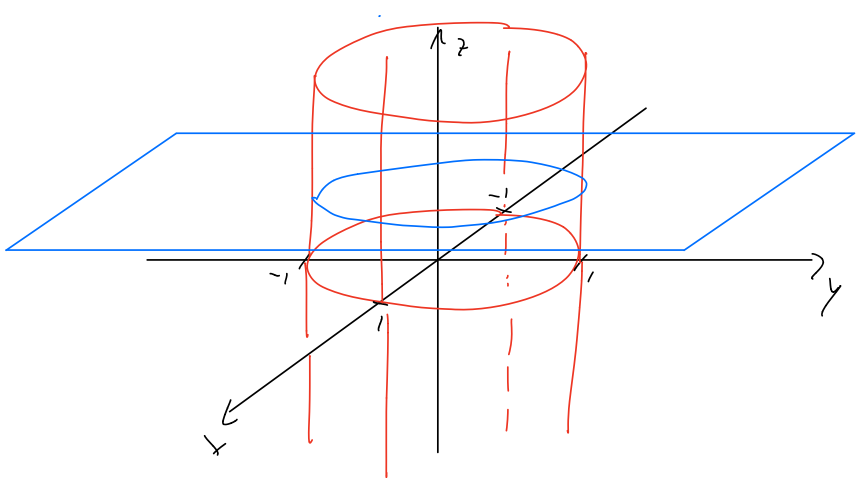


Among all points in this cylinder, we want those with z-coordinate equal to 3:



So we obtain a circle of radius 1 parallel to the xy-plane and at "height" $z=3$.

Observe that $z=3$ gives a surface that is a plane parallel to the xy-plane and at height $z=3$, since it corresponds to the surface $\{(x,y,z) \mid z=3\}$. Thus, $x^2+y^2=1$ and $z=3$ is the intersection of the cylinder $x^2+y^2=1$ and the plane $z=3$.

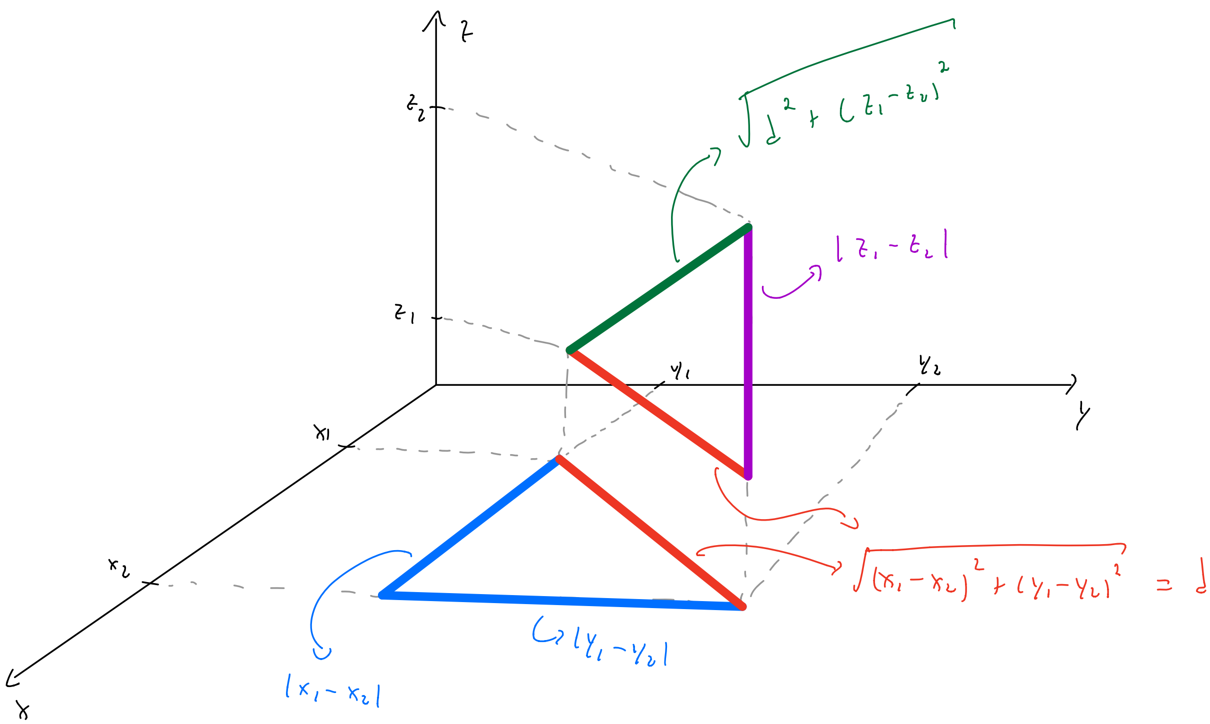


Distance and spheres

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, denoted $|P_1P_2|$, is given by

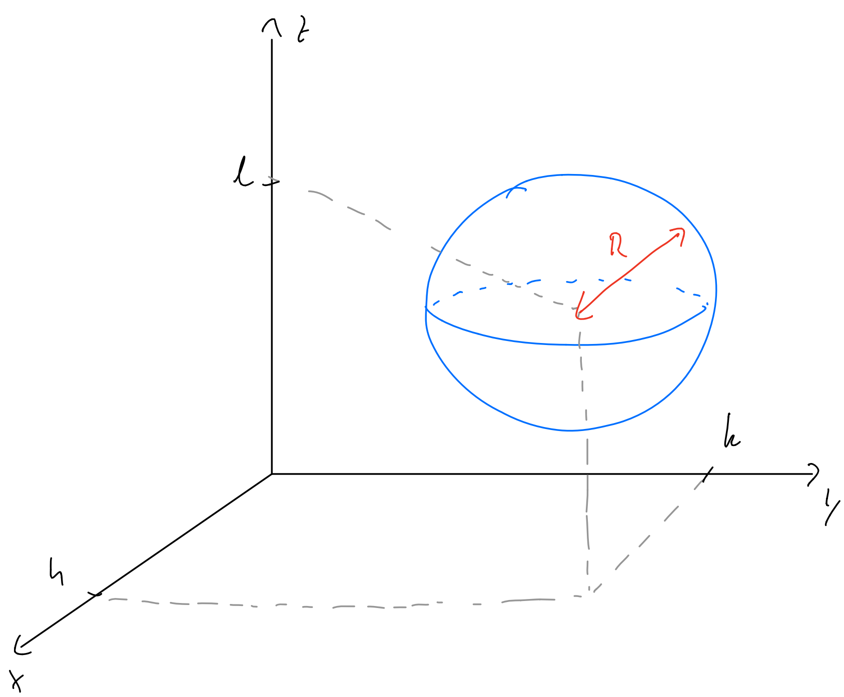
$$|P_1P_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}.$$

This follows by applying the Pythagorean theorem to the triangle in the picture:

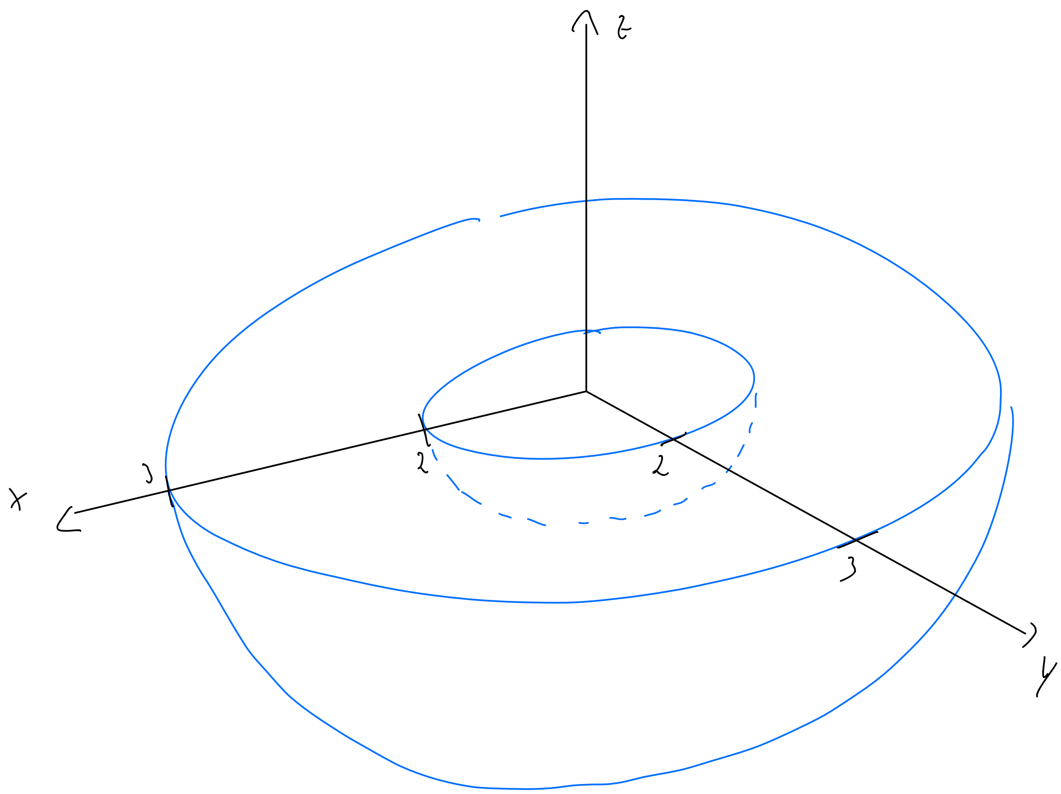


A sphere of radius R in $\mathbb{R}^3$ centered at (h,k,l) is the set of all points in $\mathbb{R}^3$ whose distance to (h,k,l) equals R. From the distance formula, we obtain that the equation for a sphere of radius R and center at (h,k,l) is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = R^2.$$



Ex: The equations $4 \leq x^2 + y^2 + z^2 \leq 9$ and $z \leq 0$ correspond to the region between the spheres $x^2 + y^2 + z^2 = 2$, $x^2 + y^2 + z^2 = 3$, and below the xy-plane (a half avocado without the pit):



__Ex:__ $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$

is the equation of a sphere of radius $\sqrt{8}$ and center $(-2, 3, -1)$. To see this, complete the squares:

$$x^2 + 4x = x^2 + 4x + 4 - 4 = (x+2)^2 - 4$$

$$y^2 - 6y = y^2 - 6y + 9 - 9 = (y-3)^2 - 9$$

$$z^2 + 2z = z^2 + 2z + 1 - 1 = (z+1)^2 - 1$$

Adding

$$x^2 + 4x + y^2 - 6y + z^2 + 2z = (x+2)^2 + (y-3)^2 + (z+1)^2 - 14$$

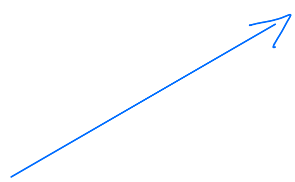
$$\parallel$$

$$-6$$

$$(x+2)^2 + (y-3)^2 + (z+1)^2 = 8 = (\sqrt{8})^2$$

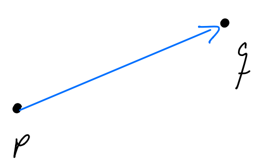
Vectors

A vector is a quantity that has both a magnitude and a direction. Examples include the physical quantities displacement and force.



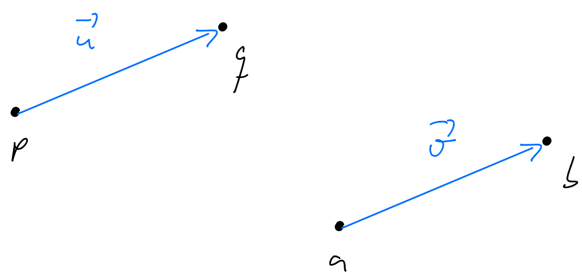
A vector: the arrow indicates the direction and the length the magnitude.

Given two points p and q, the vector with initial point at p (the tail) and end point at q (the tip) is denoted $\overrightarrow{pq}$.



For example, if a particle moves from a point a to a point b, its displacement is the vector $\overrightarrow{ab}$.

We denote vectors by letters with an arrow on top, e.g., $\vec{u}$, $\vec{v}$, etc. For example, labeling $\vec{v}$ the vector from p to q, we write $\vec{v} = \overrightarrow{pq}$. Two vectors $\vec{v}$ and $\vec{u}$ are equal, $\vec{u} = \vec{v}$, if they have the same magnitude and direction (even if they have different initial and end points).



$\vec{u}$ and $\vec{v}$ have the same direction and magnitude, so $\vec{u} = \vec{v}$.

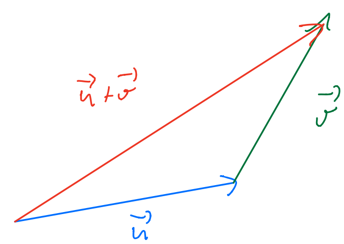
The zero vector, denoted by $\vec{0}$, has zero length. It is the only vector with no direction.

Operations with vectors

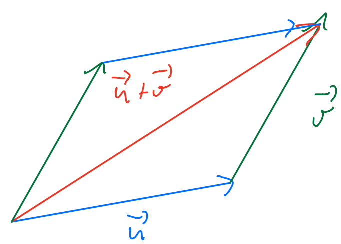
We will now define operations that allow us to add and subtract vectors and multiply them by numbers. The term scalar means a real number.

Let $\vec{u}, \vec{v}$ be vectors and c a scalar. We define:

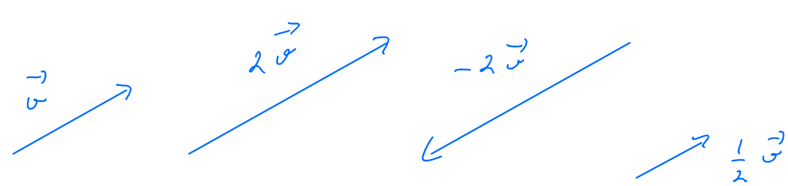
Vector addition. The sum $\vec{u}+\vec{v}$ is the vector obtained by the triangle law:



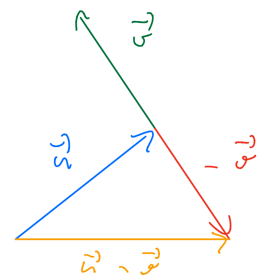
Observe that $\vec{u}+\vec{v} = \vec{v}+\vec{u}$



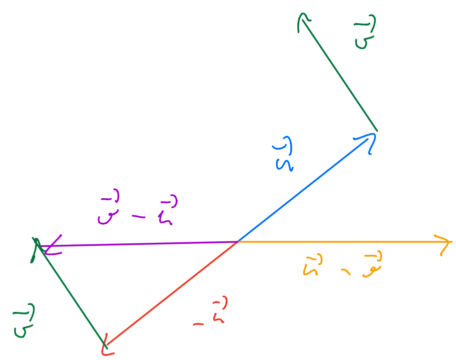
Scalar multiplication. The scalar multiplication of c and $\vec{v}$ is the vector $c\vec{v}$ whose length is $|c|$ times the length of $\vec{v}$. $c\vec{v}$ points in the direction of $\vec{v}$ if $c>0$, in the opposite direction of $\vec{v}$ if $c<0$, and $c\vec{v} = \vec{0}$ if $c=0$.



Vector difference. The difference or subtraction of two vectors $\vec{u} - \vec{v}$ is the vector defined by $\vec{u} + (-\vec{v})$.



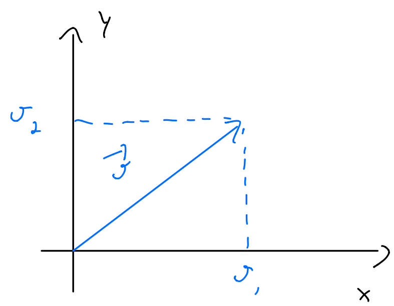
Observe that $\vec{u} - \vec{v} = -(\vec{v} - \vec{u})$.

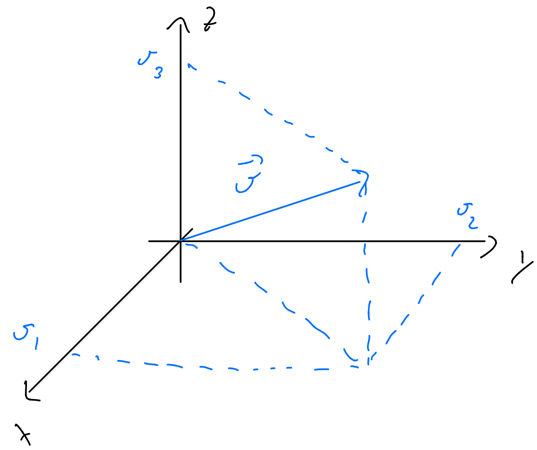


Vector components

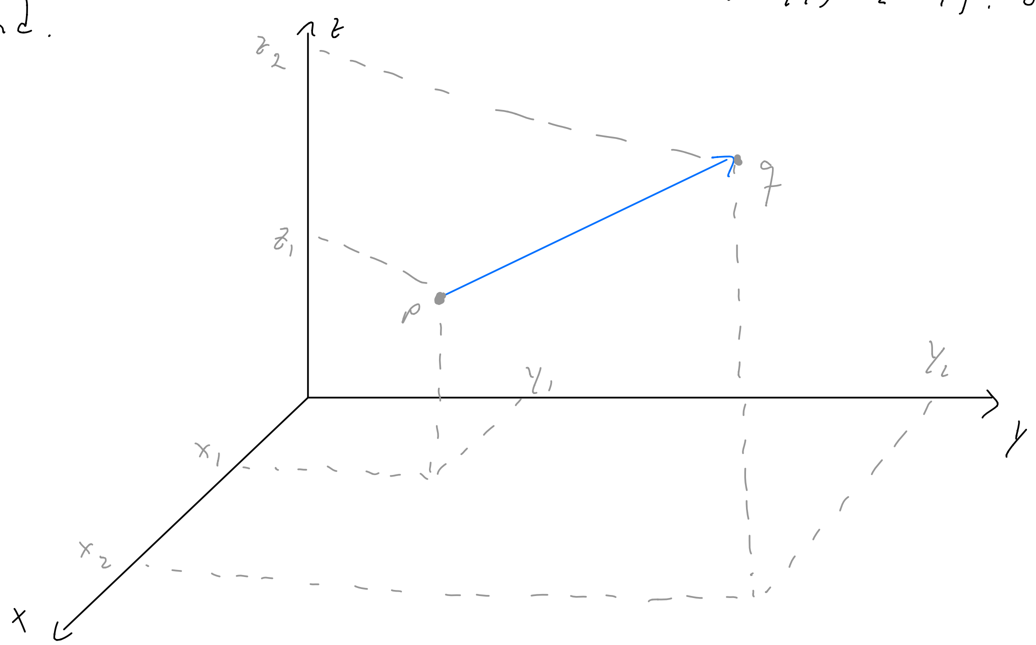
With respect to a coordinate system in 2d, a vector $\vec{v}$ with initial point at the origin $(0,0)$ is represented by $\vec{v} = (v_1, v_2)$. v_1 and v_2 are the coordinates of $\vec{v}$, v_1 is the x-coordinate and v_2 the y-coordinate.

Similarly, in 3d, a vector $\vec{v}$ with initial point at $(0,0,0)$ is represented by $\vec{v} = (v_1, v_2, v_3)$. Sometimes we can also write $\langle v_1, v_2 \rangle$, $\langle v_1, v_2, v_3 \rangle$.

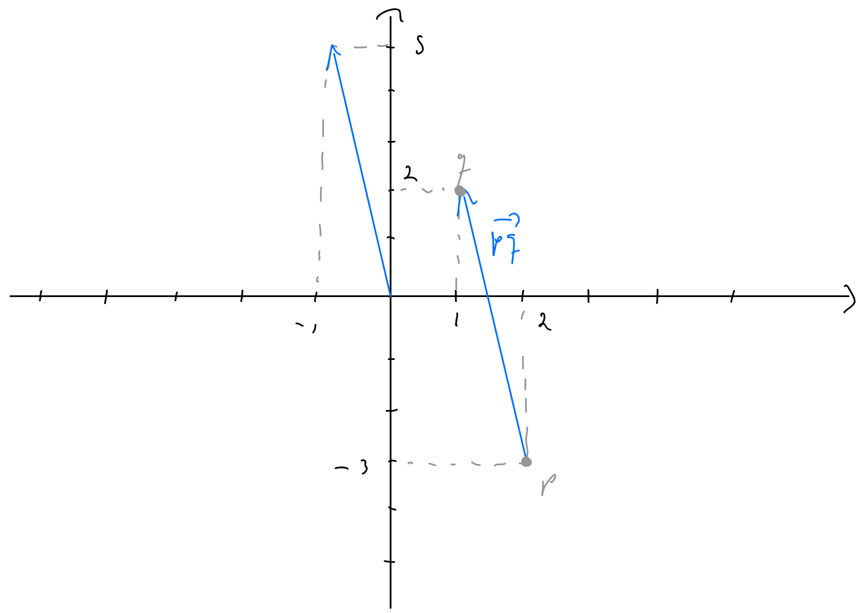




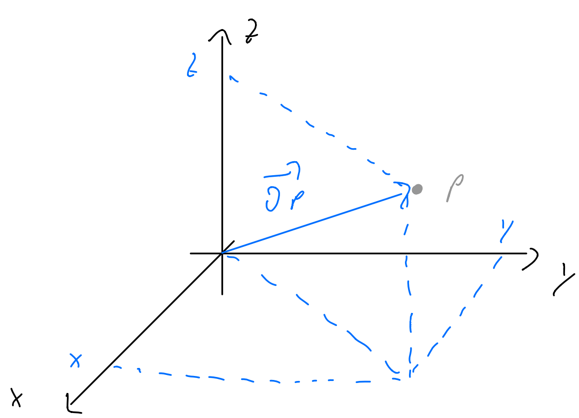
Given points $p(x_1, y_1, z_1)$ and $q(x_2, y_2, z_2)$, the vector $\vec{pq}$ is represented by $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Similarly in 2d.



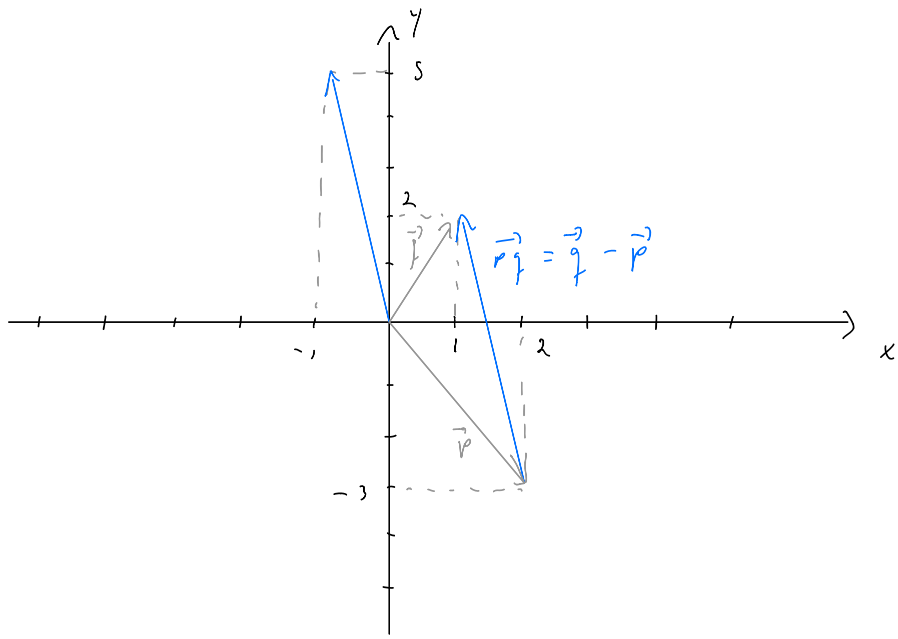
__Ex:__ Given $p(2,-3)$, $q(1,2)$, $\overrightarrow{pq} = (1-2, 2-(-3)) = (-1, 5)$



We will often identify a point $p(x,y,z)$ with its position vector, i.e., with the vector $\overrightarrow{Op}$, where O is the origin. I.e., we think of points in space as vectors with tail at the origin. Given a point p, we write $\vec{p}$ for the vector $\overrightarrow{Op}$.



It then follows that $\vec{pq} = \vec{q} - \vec{p}$ (and not $\vec{p} - \vec{q}$)



The magnitude, or length, or norm of a vector $\vec{v}$ is denoted by $|\vec{v}|$ or $\|\vec{v}\|$ and given by (Pythagorean theorem)

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2} \quad \text{or} \quad |\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

if $\vec{v} = (v_1, v_2)$ or $\vec{v} = (v_1, v_2, v_3)$, respectively. Observe that $|\vec{v}|$ is a scalar.

Addition, subtraction, and scalar multiplication of vectors can be computed componentwise, i.e., if $\vec{v} = (v_1, v_2, v_3)$, $\vec{u} = (u_1, u_2, u_3)$, and $c \in \mathbb{R}$, then

$$\vec{v} + \vec{u} = (v_1 + u_1, v_2 + u_2, v_3 + u_3),$$

$$\vec{v} - \vec{u} = (v_1 - u_1, v_2 - u_2, v_3 - u_3),$$

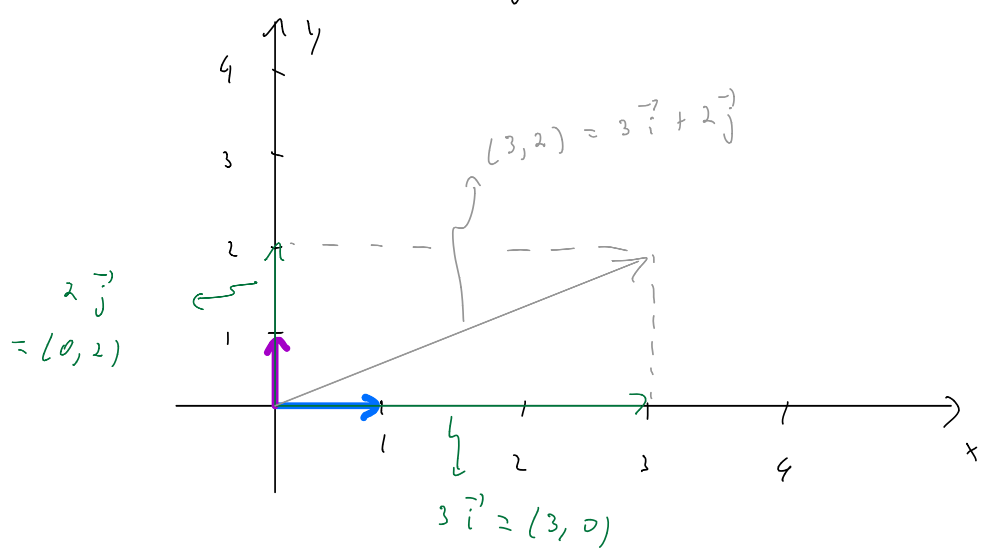
$$c\vec{v} = (cv_1, cv_2, cv_3).$$ Similarly in 2d.

The vectors $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, and $\vec{k}=(0,0,1)$ are called the standard basis vectors or canonical vectors. Any vector can be expressed in terms of them:

$$\vec{v} = (v_1, v_2, v_3) = v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1)$$

$$= v_1\vec{i} + v_2\vec{j} + v_3\vec{k}.$$

In 2d we write $\vec{i}=(1,0)$, $\vec{j}=(0,1)$.



A unit vector is a vector of length 1. For example, $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors,

$$|\vec{i}| = \sqrt{1^2 + 0^2 + 0^2} = 1.$$

Given $\vec{v} \neq \vec{0}$, a unit vector in the same direction as $\vec{v}$ is given by

$$\vec{u} = \frac{1}{|\vec{v}|}\,\vec{v}.$$

To see this, first note that

$$|c\,\vec{u}| = \underbrace{|c|}_{\text{absolute value of } c}\;\overbrace{|\vec{u}|}^{\text{norm of } \vec{u}}.$$

Then, with $c = \frac{1}{|\vec{v}|}$ ($\frac{1}{|\vec{v}|}$ is a scalar), we have

$$|\vec{u}| = \left|\frac{1}{|\vec{v}|}\,\vec{v}\right| = \frac{1}{|\vec{v}|}\,|\vec{v}| = 1.$$

Properties of vectors

An n-dimensional vector is a order n-tuple

$$\vec{v} = (v_1, v_2, \dots, v_n).$$

The set of all n-dimensional vectors is denoted V_n, which we identify with

$$\mathbb{R}^n = \overbrace{\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}}^{n \text{ times}} = \left\{ (x_1, x_2, \dots, x_n) \;\middle|\; x_i \in \mathbb{R},\ i = 1, \dots, n \right\}$$

by considering points in $\mathbb{R}^n$ as vectors as before. Addition, subtraction, and scalar multiplication of n-d vectors are defined componentwise, as above. $|\vec{v}|$ is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Properties If $\vec{v}$, $\vec{u}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ and c, d are scalars, then

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(ii) $\vec{u} + \vec{0} = \vec{u}$

(iii) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

(iv) $\vec{u} + (-\vec{u}) = \vec{u} - \vec{u} = \vec{0}$

(v) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

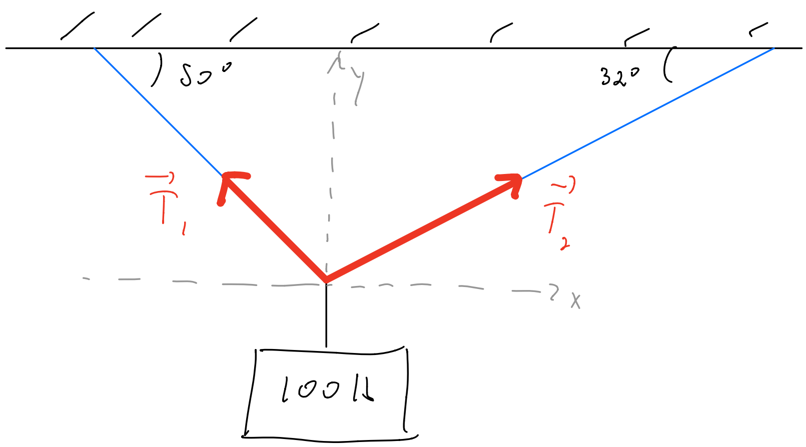
(vi) $c(d\vec{u}) = (cd)\vec{u}$

(vii) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

(viii) $1\vec{u} = \vec{u}$

Exercise: prove these properties.

Ex: A 100 lb weight hangs from two wires as in the picture. Find the tension on the wires.



From basic trigonometry:

$$\vec{T}_1 = -|\vec{T}_1|\cos 50°\,\vec{i} + |\vec{T}_1|\sin 50°\,\vec{j}$$

$$\vec{T}_2 = |\vec{T}_2|\cos 32°\,\vec{i} + |\vec{T}_2|\sin 32°\,\vec{j}$$

The weight is $\vec{w} = -100\,\vec{j}$, so $\vec{T}_1 + \vec{T}_2 = \vec{w}$, i.e.,

$$\left(-|\vec{T}_1|\cos 50° + |\vec{T}_2|\cos 32°\right)\vec{i} + \left(|\vec{T}_1|\sin 50° + |\vec{T}_2|\sin 32°\right)\vec{j}$$

$$= -100\,\vec{j} \implies \begin{cases} -|\vec{T}_1|\cos 50° + |\vec{T}_2|\cos 32° = 0 \\ |\vec{T}_1|\sin 50° + |\vec{T}_2|\sin 32° = -100 \end{cases}$$

solving for $|\vec{T}_1|$ and $|\vec{T}_2|$:

$$|\vec{T}_1| = 85.64\text{ lb}, \quad |\vec{T}_2| = 64.91\text{ lb}.$$

Thus

$$\vec{T}_1 = -55.05\vec{i} + 65.60\vec{j}$$

$$\vec{T}_2 = 55.05\vec{i} + 34.40\vec{j}.$$

The dot product

The dot product that we define below is a type of product of vectors

Def. If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (u_1, u_2, u_3)$, the dot product of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$ (a.k.a. scalar product or inner product and also denoted as $\langle \vec{u}, \vec{v} \rangle$) is defined as

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

Remark. Note that the dot product of two vectors is a number (and not a vector).

In 2d the definition is similar:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 .$$

Ex: Find $(1,-1) \cdot (2,3)$.

$(1,-1) \cdot (2,3) = 1 \cdot 2 + (-1) \cdot 3 = -1$.

Ex: Let $\vec{u} = (1,-1)$, $\vec{v} = (2,3)$. Find $\vec{u} \cdot \vec{v} + \vec{v}$.

This is not well-defined since $\vec{u} \cdot \vec{v}$ is a scalar and $\vec{v}$ a vector, and we can only add scalars to scalars and vectors to vectors. If we had $\vec{u} \cdot (\vec{v} + \vec{v})$ instead, then it would be well-defined, and $\vec{u} \cdot (\vec{v} + \vec{v}) = (1,-1) \cdot (4,6) = -2$.

Properties of the dot product. If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $n = 2, 3$, and $c \in \mathbb{R}$, then

(i) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

(ii) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(iii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(iv) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

(v) $\vec{0} \cdot \vec{u} = 0$

Exercise: prove these properties.

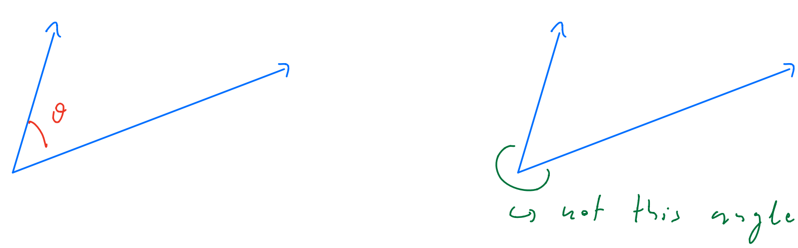
Remark. Note that in (v) above we have "different types of zeros:" on the LHS $\vec{0}$ is the zero vector, on the RHS 0 is the real number zero.

Remark. For vectors with n components, $\vec{u} \cdot \vec{v}$ is similarly defined, i.e.,

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

and the above properties still hold.

Given two vectors, the angle between them is defined as the angle between 0 and π when their tails coincide:

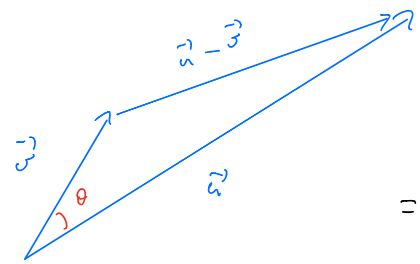


Thm. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta .$$

proof. From the law of cosines:

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\,|\vec{v}|\cos\theta$$



But

$$|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u}\cdot\vec{v},$$

giving the result. $\square$

Consequences of the theorem:

- The angle between $\vec{u}$ and $\vec{v}$ can be found from
$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} \qquad (\vec{u},\vec{v} \neq \vec{0})$$
- $\vec{u}\cdot\vec{v}$ is
  - > 0 for $0 \le \theta < \frac{\pi}{2}$
  - $= 0$ for $\theta = \frac{\pi}{2}$
  - < 0 for $\frac{\pi}{2} < \theta \le \pi$
- $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}\cdot\vec{v} = 0$.

Ex: Find the angle between $(2,1,-1)$ and $(0,2,1)$.

$$|(2,1,-1)| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{6}$$

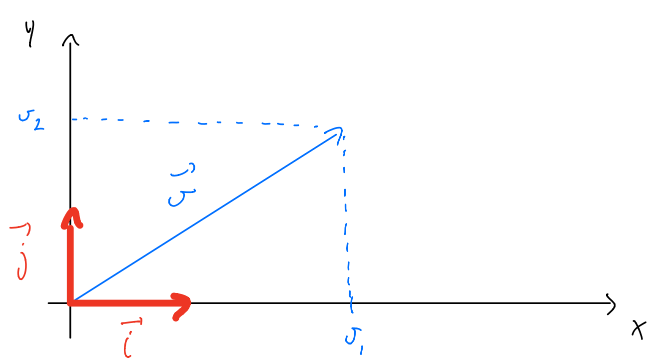
$$|(0,2,1)| = \sqrt{0^2+2^2+1} = \sqrt{5}$$

$$(2,1,-1)\cdot(0,2,1) = 0+2-1 = 1$$

$$\cos\theta = \frac{1}{\sqrt{30}}, \quad \theta = \arccos\frac{1}{\sqrt{30}} \approx 79^\circ.$$

Projections

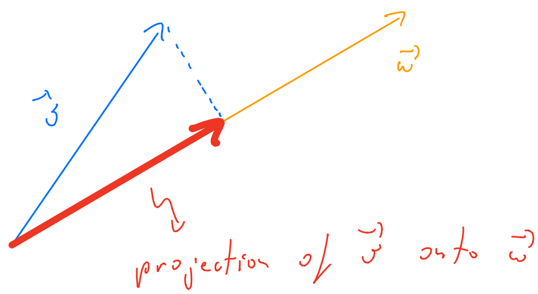
When we write $\vec{v} = v_1\vec{i} + v_2\vec{j}$, we are decomposing the vector $\vec{v}$ in terms of $\vec{i}$ and $\vec{j}$, so that $v_1\vec{i}$ and $v_2\vec{j}$ are projections of $\vec{v}$ onto $\vec{i}$ and $\vec{j}$ (or onto the x and y axis) respectively.



Observe that $v_1\vec{i} = (\vec{v}\cdot\vec{i})\,\vec{i} = \dfrac{(\vec{v}\cdot\vec{i})}{|\vec{i}|^2}\,\vec{i}$ and

$$v_2\vec{j} = \frac{\vec{v}\cdot\vec{j}}{|\vec{j}|^2}\,\vec{j}.$$

More generally, we can project a vector $\vec{v}$ onto another vector $\vec{w}$:



Def. The scalar projection of $\vec{v}$ onto $\vec{u}$, a.k.a. the component of $\vec{v}$ in the direction $\vec{u}$, is

$$\operatorname{comp}_{\vec{u}} \vec{v} := \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}.$$

The vector projection of $\vec{v}$ onto $\vec{u}$ is

$$\operatorname{proj}_{\vec{u}} \vec{v} = \operatorname{comp}_{\vec{u}} \vec{v} \, \frac{\vec{u}}{|\vec{u}|} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2}.$$

The cross product

We learned the dot product, which is a product between vectors that results in a scalar. Next, we will learn another type of product between vectors, whose result is another vector.

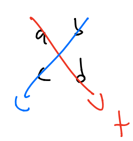
Def. Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (u_1, u_2, u_3)$. The cross product of $\vec{v}$ and $\vec{u}$, denoted by $\vec{v} \times \vec{u}$, is

$$\vec{v} \times \vec{u} := (v_2 u_3 - v_3 u_2,\ v_3 u_1 - v_1 u_3,\ v_1 u_2 - v_2 u_3)$$

A good mnemonic for the cross-product is given in terms of determinants. A determinant of order 2 is defined by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc.$$

I.e.:



A determinant of order 3 is defined by

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} := a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Then

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Ex: Find $(1,0,-1)\times(2,3,1)$.

$$(1,0,-1)\times(2,3,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}\vec{k} = 3\vec{i} - 3\vec{j} + 3\vec{k}$$

$$= (3,-3,3).$$

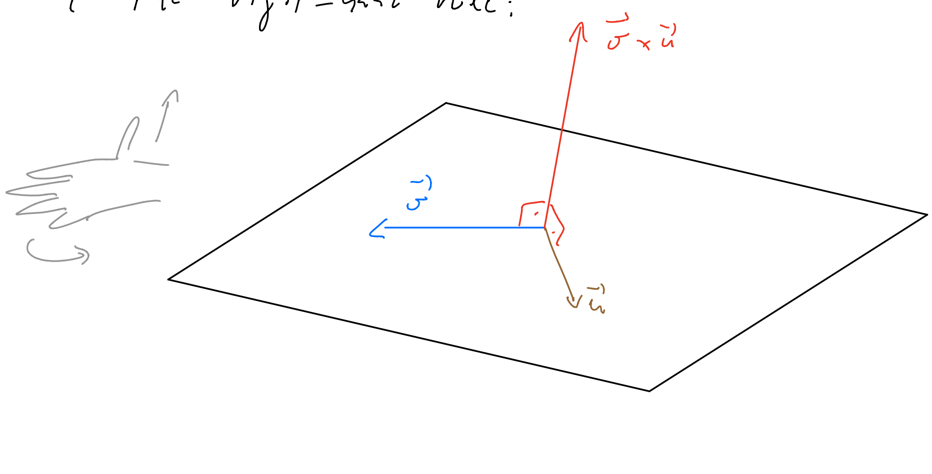
To understand the direction of $\vec{v}\times\vec{u}$, let us compute:

$$(\vec{v}\times\vec{u})\cdot\vec{v} = (v_2u_3 - v_3u_2,\ v_3u_1 - v_1u_3,\ v_1u_2 - v_2u_3)\cdot(v_1,v_2,v_3)$$

$$= \underline{v_1v_2u_3} - \underline{v_1v_3u_2} + \underline{v_2v_3u_1} - \underline{v_2v_1u_3} + \underline{v_3v_1u_2} - \underline{v_2u_3v_3} = 0.$$

So $\vec{v}\times\vec{u}$ is orthogonal to $\vec{v}$. Similarly we find that it is orthogonal to $\vec{u}$. Thus $\vec{v}\times\vec{u}$ is orthogonal to the plane containing $\vec{v}$ and $\vec{u}$.

We can verify that the direction of $\vec{v}\times\vec{u}$ is given by the right-hand rule:



Properties of the cross product. Let $\vec{v}, \vec{u}, \vec{w}$ be vectors in $\mathbb{R}^3$ (the cross product is only defined in $\mathbb{R}^3$) and c a scalar. Then:

(i) $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$ (keep the order!)

(ii) $(c\vec{v}) \times \vec{u} = c(\vec{v} \times \vec{u}) = \vec{v} \times (c\vec{u})$

(iii) $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$

(iv) $(\vec{v} + \vec{u}) \times \vec{w} = \vec{v} \times \vec{w} + \vec{u} \times \vec{w}$

(v) $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w}$ (triple product)

(vi) $\vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}$. In general:

$$\vec{v} \times (\vec{u} \times \vec{w}) \neq (\vec{v} \times \vec{u}) \times \vec{w}.$$

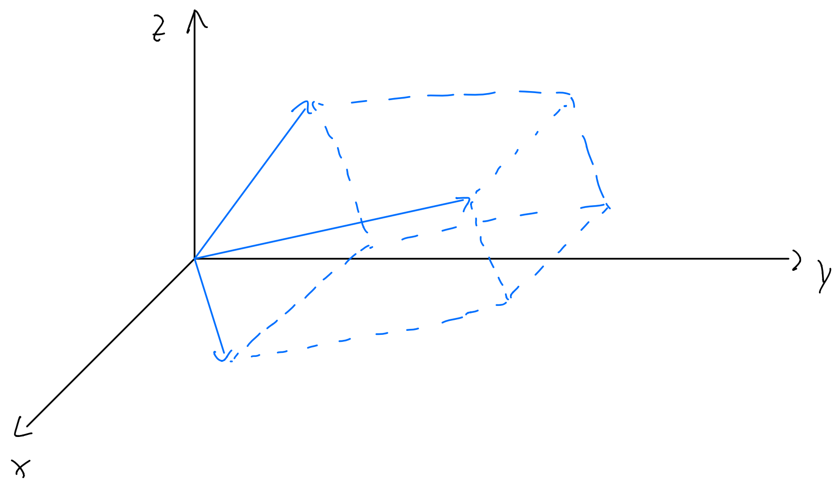
Geometric properties of the cross product. Let $\vec{v}, \vec{u}, \vec{w}$ be vectors in $\mathbb{R}^3$.

- If θ is the angle between $\vec{v}$ and $\vec{u}$, then

$$|\vec{v} \times \vec{u}| = |\vec{v}||\vec{u}| \sin\theta.$$

- $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{v} \times \vec{u} = \vec{0}$ for $\vec{u}, \vec{v} \neq \vec{0}$.

- The volume of the parallelepiped determined by $\vec{v}, \vec{u}, \vec{w}$ is $|\vec{v} \cdot (\vec{u} \times \vec{w})|$.


z
y
x

Equations of lines and planes

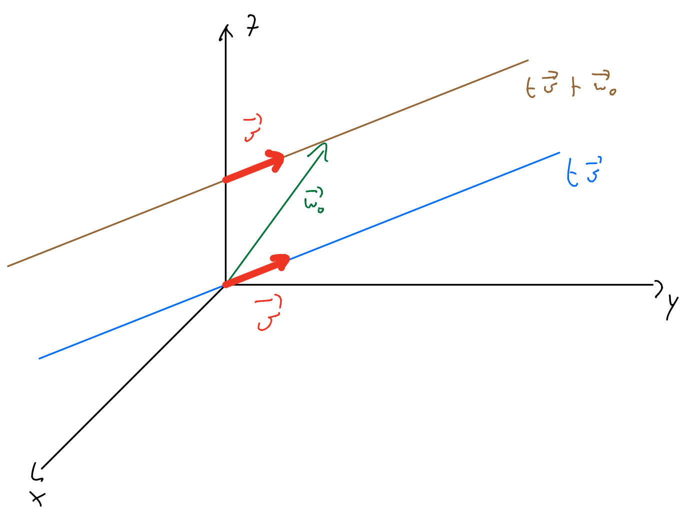
Lines

A line in $\mathbb{R}^3$ is determined by a point and a direction. For example, given $\vec{v}$, the set of points

$$\vec{w} = t\vec{v}, \quad t \in \mathbb{R},$$

is a line in the direction (parallel to) $\vec{v}$ passing through the origin. More generally, the equation of of a line in the direction $\vec{v}$ and passing throug the point $w_0 = (x_0, y_0, z_0)$ is, in vector form,

$$\vec{w} = t\vec{v} + \vec{w}_0.$$



Writing $\vec{w} = (x, y, z)$ and $\vec{v} = (a, b, c)$, we can write the equations for each component of the line:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + at,$$

which are known as parametric equations of a line with direction $\vec{v}$ and through the point $\vec{w_0}$.

Eliminating t in each one of the parametric equations we also have

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which are known as the symetric equations of a line in the $\vec{v}$ direction and through $\vec{w_0}$. If a component of $\vec{v}$ is zero, say, $a = 0$, we write

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Ex: Find the equation of the line through the points $(-8, 1, 4)$, $(3, -2, 4)$. Does it intersect the xz-plane?

We can find $\vec{v}$ by

$$\vec{v} = (3, -2, 4) - (-8, 1, 4) = (11, -3, 0)$$

Thus

$$\vec{w} = t(11, -3, 0) + (-8, 1, 4).$$

The same line is described by $t(11, -3, 0) + (3, -2, 4)$. Or yet $t(\frac{11}{3}, -1, 0) + (-8, 1, 4)$, etc. In parametric and symmetric forms

$$\begin{aligned} x &= 11t - 8 \\ y &= -3t + 1 \\ z &= 4 \end{aligned} \quad , \quad \frac{x + 8}{11} = \frac{y - 1}{-3}, \quad z = 4$$

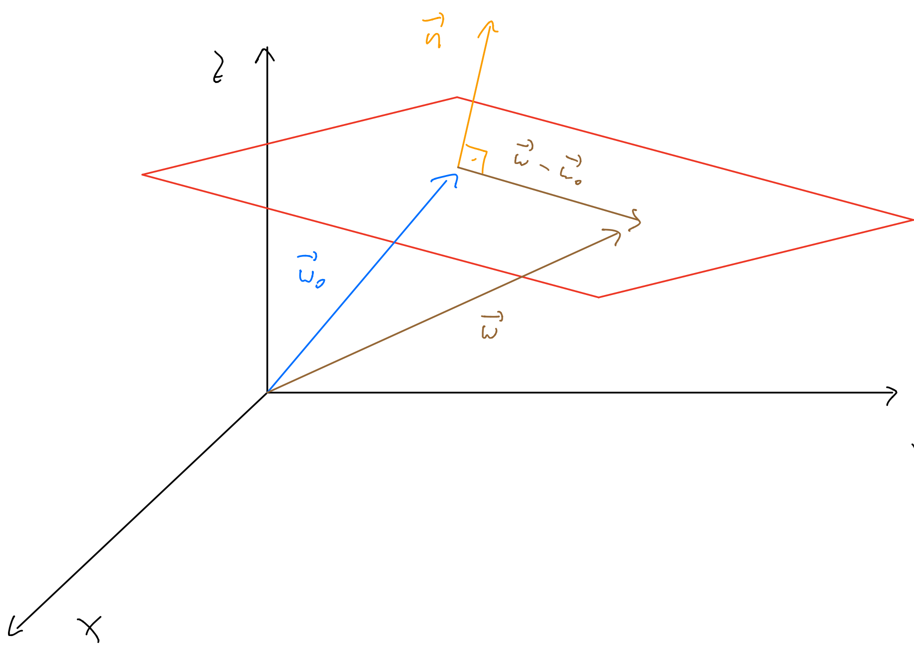
Intersection with the xz-plane happens when $y=0$, so

$$\frac{x+8}{11} = \frac{0-1}{-3} \Rightarrow x = \frac{11}{3} - 8, \quad t = 4,$$

so the intersection happens at the point $\left(\frac{11}{3}, 0, 4\right)$.

Planes

A plane in $\mathbb{R}^3$ is determined by a point and a vector orthogonal to the plane. Consider a plane containing the point $\vec{w}_0$ and that is normal to $\vec{n}$.



If $\vec{w}$ represents any other point on the plane, then $\vec{w} - \vec{w}_0$ is parallel to the plane. So, $\vec{w} - \vec{w}_0$ is orthogonal to $\vec{n}$, hence

$$(\vec{w} - \vec{w}_0) \cdot \vec{n} = 0 .$$

The plane is formed by all those $\vec{w}$ that satisfy the above equation, known as the vector equation of the plane.

If $\vec{w} = (x, y, z)$, $\vec{w}_0 = (x_0, y_0, z_0)$, and $\vec{n} = (a, b, c)$, the above can be written as

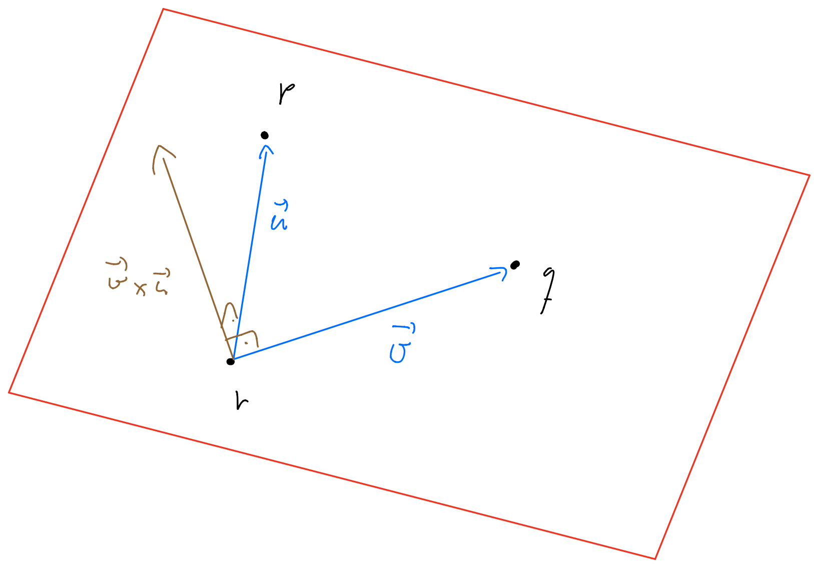
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

known as the scalar equation of the plane. We can also write

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$, known as the linear equation of the plane.

Ex: Find an equation for the plane through $(2,1,2)$, $(3,-8,6)$, $(-2,-3,1)$.



We find two vectors on the plane by

$$\vec{v} = (3,-8,6) - (2,1,2) = (1,-9,4)$$

$$\vec{u} = (-2,-3,1) - (2,1,2) = (-4,-4,-1).$$

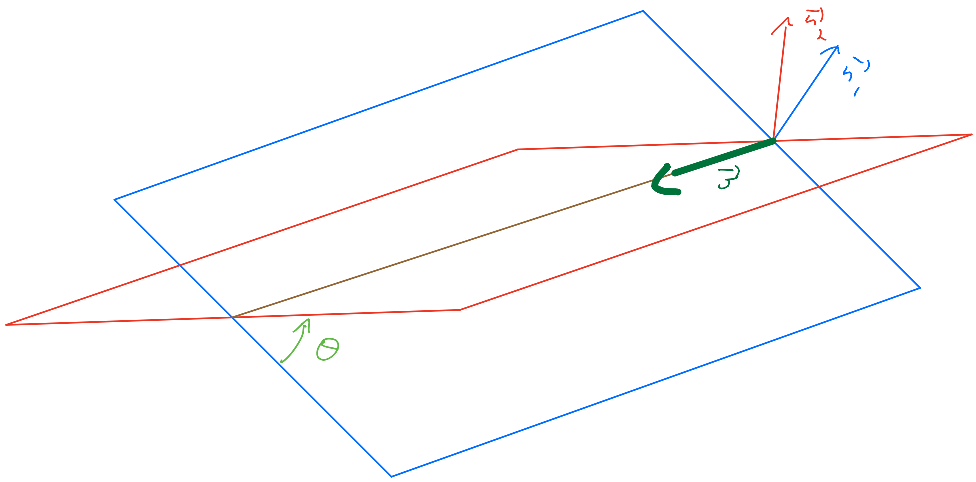
$\vec{v} \times \vec{u} = (25,-15,-40)$ is thus orthogonal to the plane.

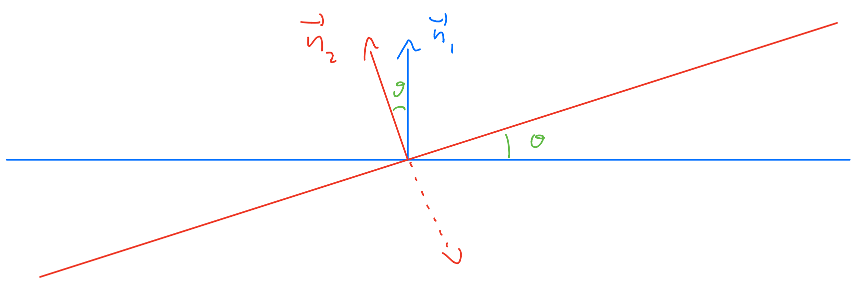
Thus, with $\vec{v}_0 = (2,1,2)$,

$$25(x-2) - 15(y-1) - 40(z-2) = 0.$$

__Ex:__ Other typical problems:

- Angle between planes

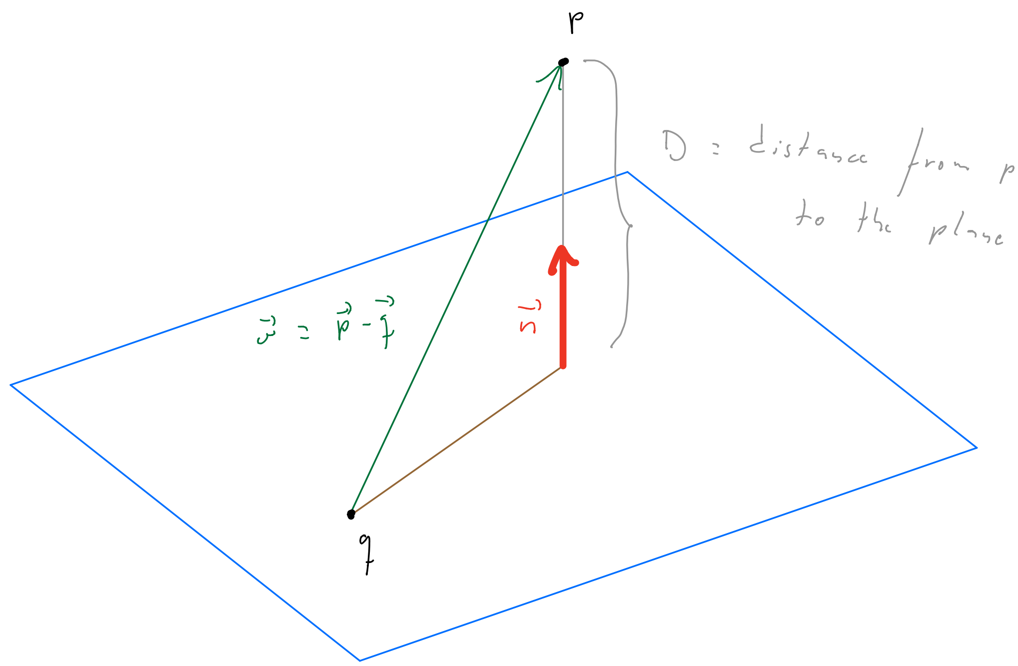




angle between planes: acute ($0 \le \theta \le \frac{\pi}{2}$) angle between their normals.

- line determined by the intersection of two planes. Direction of the line given by $\vec{v} = \vec{n}_1 \times \vec{n}_2$. Point on the line: set, e.g., $z=0$ and solve for x and y.

- Distance between a point and a plane



$$D = \left|\text{comp}_{\vec{n}}\, \vec{w}\right| = \frac{|\vec{n} \cdot \vec{w}|}{|\vec{n}|}, \qquad \vec{n} = (a,b,c), \quad \vec{q} = (x_0, y_0, z_0), \quad \vec{p} = (x, y, z)$$

$$= \frac{|(a,b,c) \cdot (x - x_0,\ y - y_0,\ z - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax + bx + cx - \overbrace{(ax_0 + bx_0 + cx_0)}^{= -d}|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax + bx + cx + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Cylinders and quadric surfaces

A quadric surface is the set of (x,y,z) in $\mathbb{R}^3$ that satisfies a second degree equation. The most general quadric surface is given by

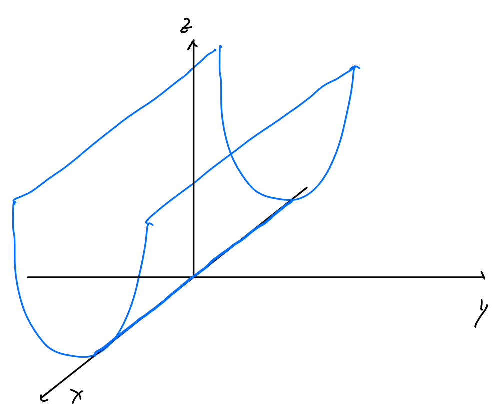
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, \ldots, J$ are constants (coefficients). E.g., the sphere

$$x^2 + y^2 + z^2 = 1$$

is a quadric surface with $A = B = C = 1$, $J = -1$, and the other coefficients equal to zero.

A surface that consists of all lines parallel to a given line (called rulings) and passing through a given plane curve is called a cylinder. E.g., $y = z^2$ is the cylinder:

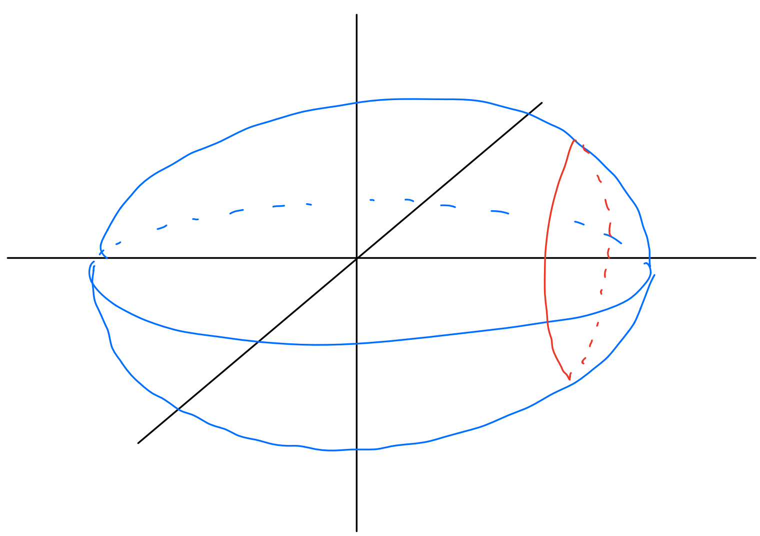


Intersections of surfaces with planes parallel to the coordinate planes are called traces. E.g., the trace obtained by intersecting $x = \text{constant}$ with the above cylinder is a parabola.

Examples of quadric surfaces

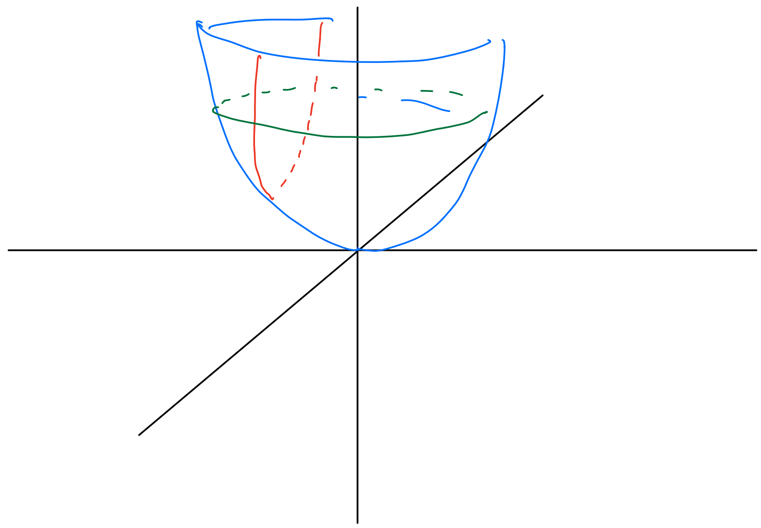
Ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



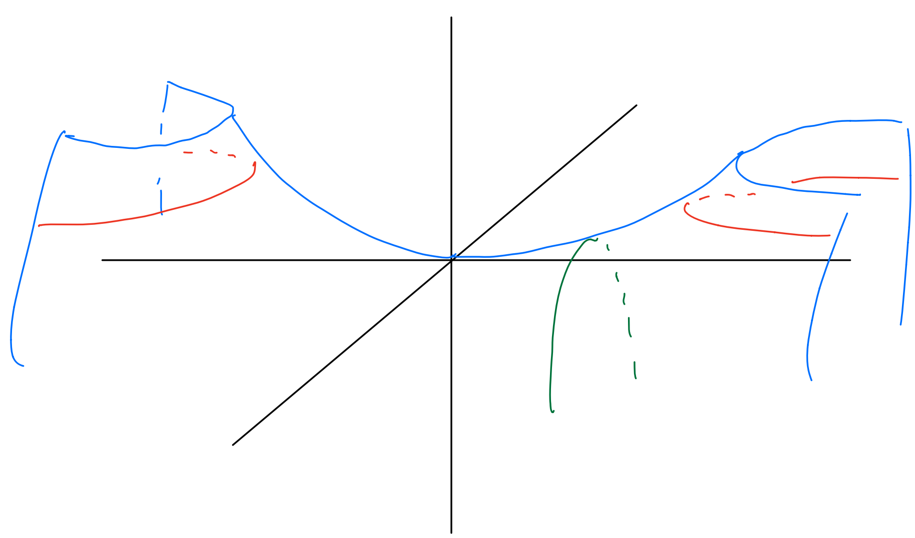
All traces are ellipses.

Elliptic paraboloid. $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



Horizontal traces are ellipses, vertical traces are parabolas

Hyperbolic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

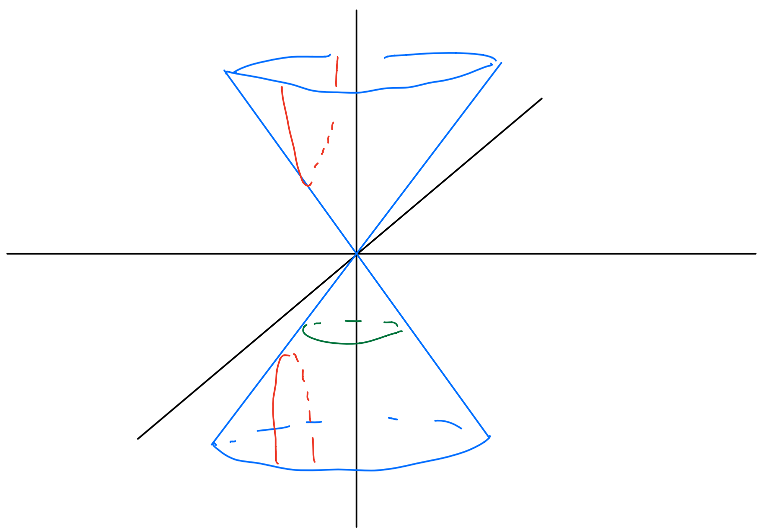


$(c>0)$

Horizontal traces are hyperbolas, vertical traces are parabolas

Cone

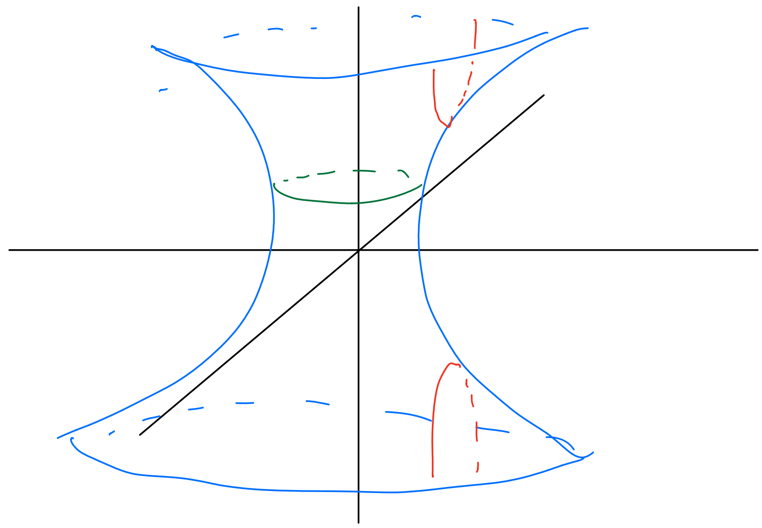
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



Horizontal traces are ellipses, vertical traces are hyperbolas or lines

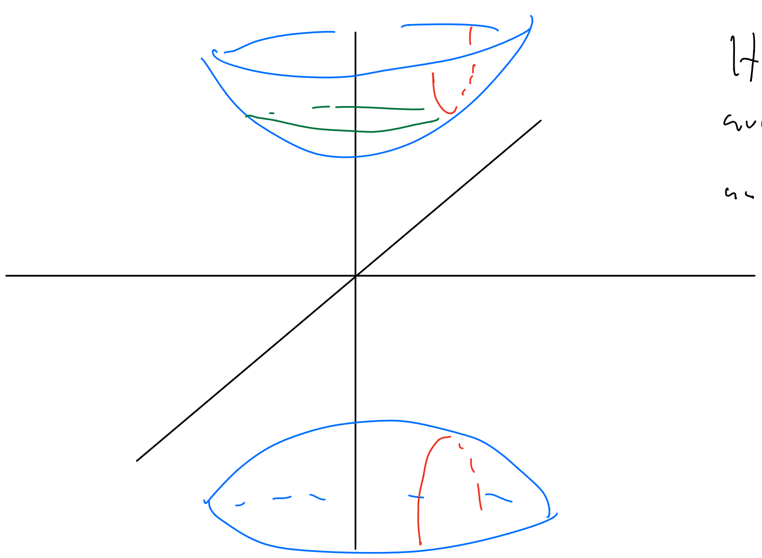
Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



Horizontal traces are ellipses, vertical traces are hyperbolas

Hyperboloid of two sheets. $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



Horizontal traces are ellipses (or empty) and vertical traces are hyperbolas.

Vector functions and space curves

A vector-valued function or vector function is a function taking values in $\mathbb{R}^3$, i.e., whose range is $\mathbb{R}^3$. We will deal first with vector-valued functions whose domain is a subset of $\mathbb{R}$. When not stated explicitly, it is understood that the domain is always the largest set in $\mathbb{R}$ for which all the expressions defining the vector-valued function are well defined. Writing a vector-valued function as

$$\vec{r}(t) = (f(t), g(t), h(t)) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k},$$

where f, g, h are scalar functions (i.e., real valued functions), we call f, g, h the component functions of the vector-valued function $\vec{r}$.

Ex: If $\vec{r}(t) = \left(t^2, \frac{1}{t}, \sqrt{t+2}\right)$, the component functions are $f(t) = t^2$, $g(t) = \frac{1}{t}$, $h(t) = \sqrt{t+2}$, whose domains are $(-\infty,\infty)$, $(-\infty,0)\cup(0,\infty)$, $[-2,\infty)$, respectively. Thus the domain of $\vec{r}(t)$ is $[-2,0)\cup(0,\infty)$.

Limits and continuity

Def. If $\vec{r}(t) = (f(t), g(t), h(t))$, the limit

$$\lim_{t\to a} \vec{r}(t)$$

is defined as

$$\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right)$$

provided the limits on the RHS exist. We similarly define limits $t\to a^+$, $t\to a^-$. $\vec{r}(t)$ is continuous at a if

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a).$$

Ex: let $f(t) = \begin{cases} 1, & t \neq 0 \\ 0, & t = 0 \end{cases}$, $g(t) = t$, $h(t) = \dfrac{1}{t^2-1}$

$\vec{r}(t) = (f(t), g(t), h(t))$. Find $\lim_{t\to 0} \vec{r}(t)$. At which points is $\vec{r}(t)$ continuous?

Since $\lim_{t\to 0} f(t) = 1$, $\lim_{t\to 0} g(t) = 0$, $\lim_{t\to 0} h(t) = -1$,

we have $\lim_{t\to 0} \vec{r}(t) = (1, 0, -1)$. f is not continuous at 0, and h at ± 1. Thus, $\vec{r}$ is continuous everywhere except at $t=0$, $t=-1$, and $t=1$.

Space curves

If f, g, h are continuous functions defined on an interval I, the set of points

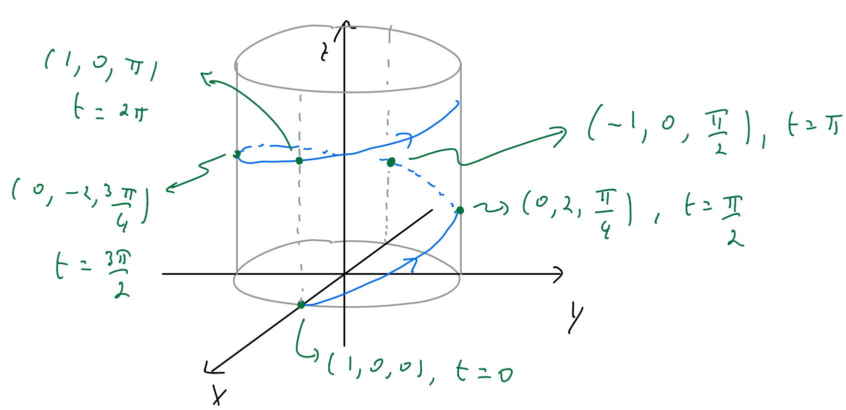
$$x = f(t),\quad y = g(t),\quad z = h(t),\qquad t \in I,$$

defines a curve in $\mathbb{R}^3$ called a space curve, whose equations above are called the parametric equations of the curve. t is called the parameter. If $\vec{r}(t) = (f(t), g(t), h(t))$, the curve given by the parametric equations in f, g, h is the space curve corresponding to $\vec{r}$.

Ex: Sketch the curve given by

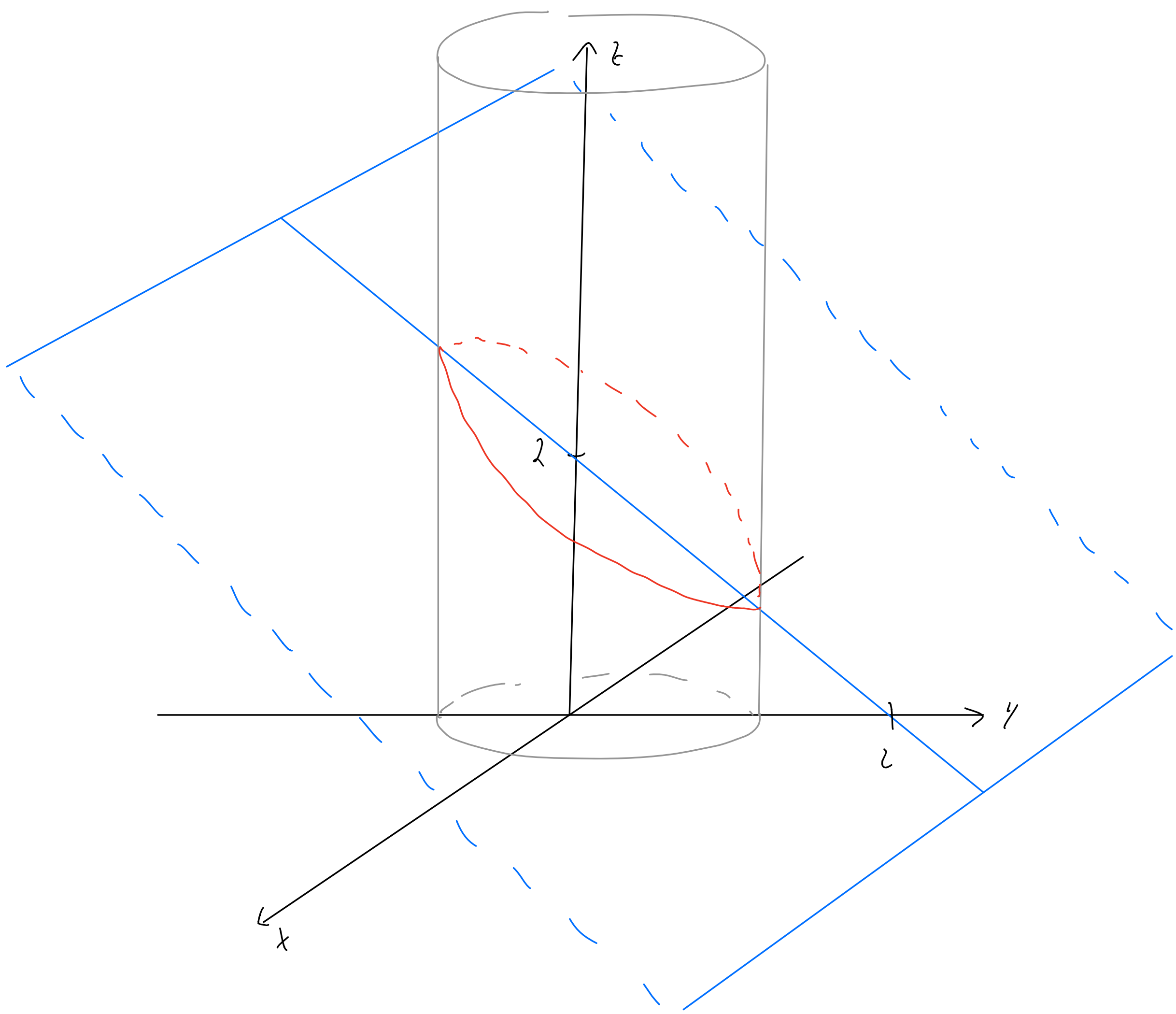
$$\vec{r}(t) = \left(\cos t,\ 2\sin t,\ \frac{t}{2}\right),\qquad t \geq 0.$$

$(\cos t, 2\sin t)$ defines an ellipse on the xy-plane. As t varies, we get a curve whose projection on the xy-plane is this ellipse



When drawing space curves, we indicate the direction of increasing t with an arrow, as above.

<u>EX</u>: Find a vector function whose curve represents the curve of the intersection of $x^2+y^2=1$ with $y+z=2$.



The projection of the curve on the xy-plane is $x^2+y^2=1$. So we can parametrize

$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

Since $z = 2 - y$, we have $z = 2 - \sin t$. Thus

$$\vec{r}(t) = (\cos t, \sin t, 2 - \sin t).$$

Derivatives and integrals of vector functions

Def. Given a vector function $\vec{r}$, its derivative, denoted $\vec{r}\,'$ or $\frac{d\vec{r}}{dt}$, is the vector function given by

$$\vec{r}\,'(t) := \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h},$$

provided the limit exists, in which case we say that $\vec{r}$ is differentiable.

If $\vec{r} = (f, g, h)$, then

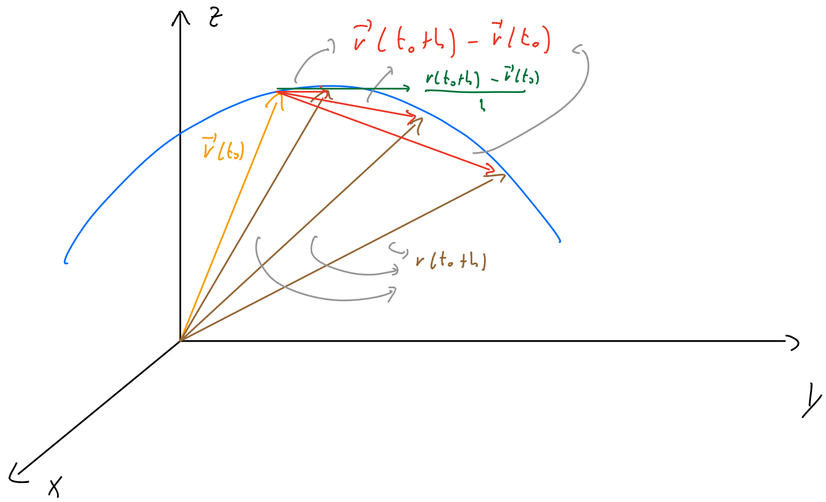
$$\vec{r}\,'(t) = (f'(t), g'(t), h'(t)).$$

Exercise: prove this formula.

Ex: Find $\vec{r}\,'(t)$ if $\vec{r}(t) = (t^2, \sin t, 7)$.

$r'(t) = (2t, \cos t, 0)$.

When $\vec{r}'(t_0) \neq \vec{0}$, then $\vec{r}'(t_0)$ is a vector that is tangent to the curve $\vec{r}(t)$ at $\vec{r}(t_0)$.



The vector

$$T(t) := \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

is a unit tangent vector to the curve ($\vec{r}'(t) \neq \vec{0}$).

Properties. Let $\vec{r}(t)$, $\vec{p}(t)$ and $f(t)$ be differentiable and c a constant. Then

(i) $(\vec{r}(t) + \vec{p}(t))' = \vec{r}'(t) + p'(t)$. (ii) $(c\,\vec{r}(t))' = c\,\vec{r}'(t)$

(iii) $(f(t)\,\vec{r}(t))' = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}'(t)$

(iv) $(\vec{r}(t) \cdot \vec{p}(t))' = \vec{r}'(t) \cdot \vec{p}(t) + \vec{r}(t) \cdot \vec{p}'(t)$

(v) $(\vec{r}(t) \times \vec{p}(t))' = \vec{r}'(t) \times \vec{p}(t) + \vec{r}(t) \times \vec{p}'(t)$ (order matters!)

(vi) $(\vec{r}(f(t)))' = \vec{r}'(f(t))\,f'(t)$. (chain rule)

Exercise: proof these properties.

Suppose that $\vec{r}(t)$ is such that $|\vec{r}(t)|$ is constant, e.g., $\vec{r}(t) = (\cos t, \sin t, 0)$, so

$$|\vec{r}(t)| = \sqrt{(\cos t)^2 + (\sin t)^2 + 0^2} = 1 \text{ for any } t.$$

Then $\vec{r}\,'(t)$ and $\vec{r}(t)$ are orthogonal:

$$|\vec{r}(t)|^2 = c \Rightarrow (|\vec{r}(t)|^2)' = 0$$

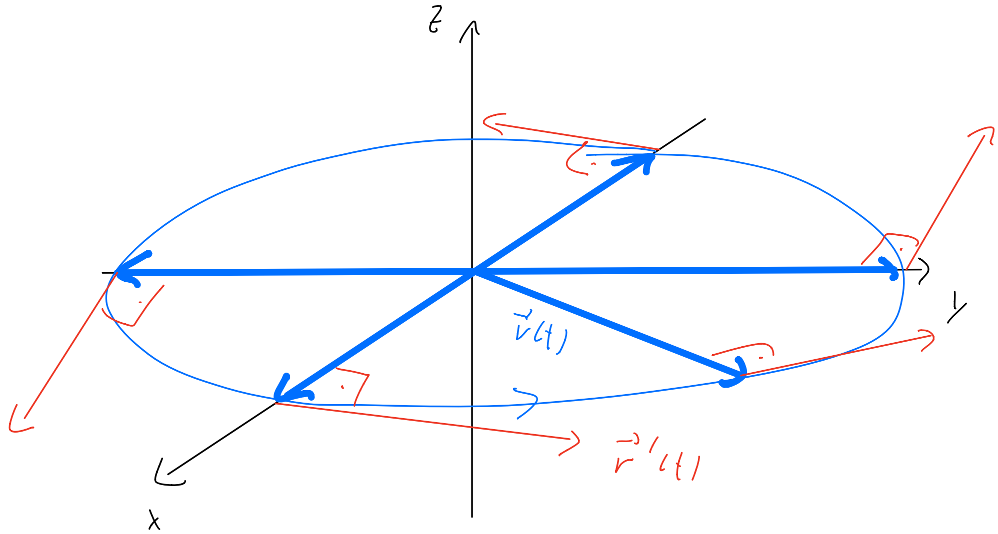
$$\parallel$$

$$(\vec{r}(t)\cdot\vec{r}(t))' = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$$
$$= 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$$

$$\Rightarrow \vec{r}(t)\cdot\vec{r}\,'(t) = 0.$$

In the example $\vec{r}(t) = (\cos t, \sin t, 0)$, $\vec{r}\,'(t) = (-\sin t, \cos t, 0)$,

$\vec{r}(t)\cdot\vec{r}\,'(t) = -\cos t \sin t + \sin t \cos t + 0 = 0.$



Integrals

If $\vec{r}(t) = (f(t), g(t), h(t))$, we define the integral of $\vec{r}(t)$ from a to b as

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}.$$

For the (indefinite integral) $\int \vec{r}(t)\,dt$, we add a constant vector.

Ex: Find $\int \vec{r}(t)\,dt$ if $\vec{r}(t) = (t^2, 1, t)$

$$\int f(t)\,dt = \int t^2\,dt = \frac{t^3}{3} + c_1$$

$$\int g(t)\,dt = \int dt = t + c_2$$

$$\int h(t)\,dt = \int t\,dt = \frac{t^2}{2} + c_3$$

$$\int \vec{r}(t)\,dt = \left(\frac{t^3}{3}, t, \frac{t^2}{2}\right) + \underbrace{\vec{C}}_{(c_1, c_2, c_3)}.$$

The fundamental theorem of calculus also holds for vector functions:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a).$$

Arc length and curvature

Given a continuously differentiable vector-valued function $\vec{r}(t) = (f(t), g(t), h(t))$, the length L of the space curve obtained when t increases from a to b is given by

$$L = \int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$

where $x = f(t)$, $y = g(t)$, $z = h(t)$.

Ex: The length of the curve

$$\vec{r}(t) = (2\cos t, 2\sin t, -1), \quad 0 \le t \le \pi$$

is

$$\int_0^\pi |\vec{r}\,'(t)|\,dt = \int_0^\pi \sqrt{4\sin^2 t + 4\cos^2 t + 0^2}\,dt = 2\pi.$$

It is often useful to parametrize curves in such a way that one unit of the parameter corresponds to one unit of the curve's length. I.e., can we change our "units of time" such that one unit of time corresponds numerically to exactly to one unit of the curve's length? In the previous example, t varied from 0 to π but the curve's length was twice that, 2π. But if we change variables $s = 2t$ and re-express $\vec{r}(t)$ in terms of s:

$$\vec{r}(s) = \left(2\cos\frac{s}{2},\ 2\sin\frac{s}{2},\ -1\right), \quad \underbrace{0 \leq s \leq 2\pi}_{\text{since } 0 \leq \underset{s/2}{\overset{\shortparallel}{t}} \leq \pi}$$

Then, by change of variable,

$$\int_0^{\pi} \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^{2\pi} \left|\frac{d\vec{r}}{ds}\right| ds = \int_0^{2\pi} \sqrt{\sin^2\left(\frac{s}{2}\right) + \cos^2\left(\frac{s}{2}\right) + 0}\ ds = 2\pi.$$

So the length of the curve (which cannot change by using the new variable s) is 2π and the new variable also varies on an interval of same length.

A change of variables in the variable t as above is called a reparametrization of the curve. The arc length function $s(t)$ of a curve is a function that has the property that if the curve is parametrized in terms of s, then the length of the curve obtained from varying s from a to b ($b > a$) is exactly $b - a$.

The arc length is defined as

$$s(t) := \int_a^t |\vec{v}\,'(\tau)|\, d\tau .$$

so that $\dfrac{ds}{dt} = |\vec{v}\,'(t)|$. Then, if t varies from a to b, $s(t)$ varies from $s(a)$ to $s(b)$ and (note: $\frac{ds}{dt} \geq 0$)

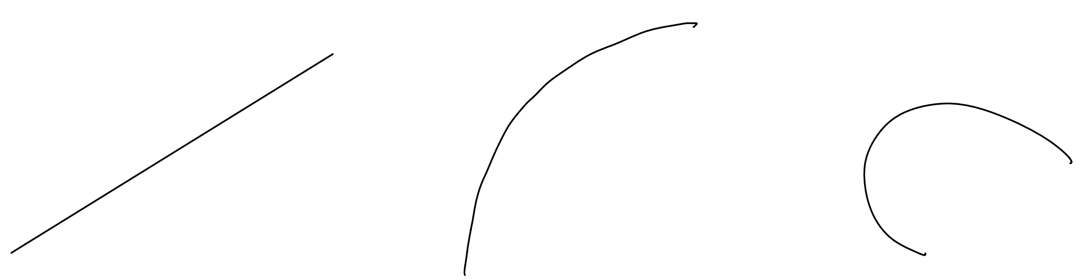
$$\int_a^b |\vec{r}\,'(t)|\, dt = \int_a^b \frac{ds}{dt}\, dt = \int_{s(a)}^{s(b)} ds = s(b) - s(a).$$

Note that we have $s = s(t)$ but we can invert it $t = t(s)$ as in the previous example.

$$\int_{s(a)}^{s(b)} f(s)\, ds = \int_a^b f(s(t))\, \frac{ds}{dt}\, dt$$

Curvature

We have an intuitive notation of the curvature of a curve:

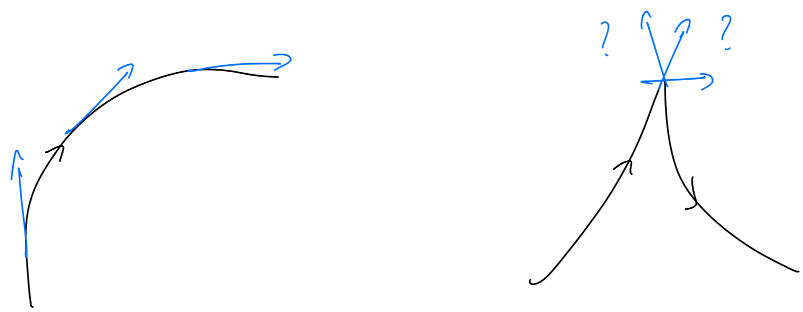


flat, no curvature   small curvature   large curvature

We will now see how to measure this mathematically.

Def. A parametrization $\vec{r}(t)$ is called smooth if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$. A curve is called smooth if it has a smooth parametrization.

A smooth curve always has a well defined tangent vector; it has no corners or cusps.



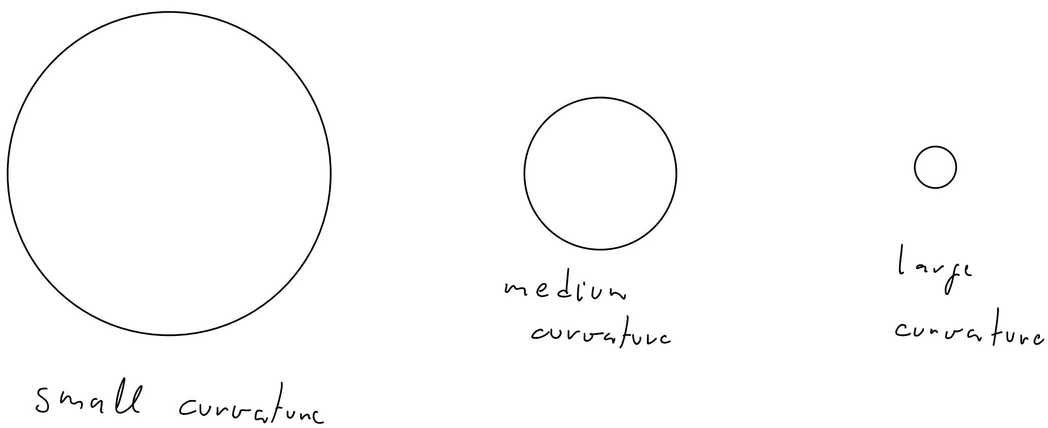
smooth　　　　　　not smooth

Def. The curvature of a curve defined by a vector function $\vec{r}(t)$ is

$$\kappa(s) = \kappa = \left| \frac{d\vec{T}}{ds} \right|$$

where $\vec{T}$ is the unit tangent vector and s the arc length.

Ex: Curvature of circles



Circle of radius R:

$$\vec{r}(t) = (R\cos t, R\sin t), \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = (-R\sin t, R\cos t), \quad |\vec{r}\,'(t)| = R$$

$$\vec{T}(t) = \frac{r'(t)}{|\vec{r}\,'(t)|} = (-\sin t, \cos t)$$

$$s(t) = \int_0^t |\vec{r}\,'(\tau)|\, d\tau = \int_0^t R\, d\tau = Rt \Rightarrow t = \frac{s}{R}$$

$$\vec{T}(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right)$$

$$\frac{d\vec{T}}{ds} = \left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right), \quad \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{R}$$

$$\kappa = \frac{1}{R}.$$

Theo. The curvature is also given by

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

(so in these formulas we can use the parameter t, i.e., we don't need to change to the arc length s).

proof.

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}.$$

For the second equality: $\vec{T} = \dfrac{\vec{r}'}{|\vec{r}'|}$, so

$$\vec{r}' = |\vec{r}'|\,\vec{T} = \frac{ds}{dt}\vec{T}, \quad \text{thus} \quad \vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'.$$

$$\vec{r}' \times \vec{r}'' = \vec{r}' \times \left(\frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}' \right)$$

$$= \frac{d^2s}{dt^2}\underbrace{\vec{r}'}_{\substack{\| \\ |\vec{r}'|\vec{T}}} \times \vec{T} + \frac{ds}{dt}\underbrace{\vec{r}'}_{\substack{\| \\ |\vec{r}'|\vec{T} = \frac{ds}{dt}\vec{T}}} \times \vec{T}'$$

$$= \frac{d^2s}{dt^2}|\vec{r}'|\underbrace{\vec{T} \times \vec{T}}_{=0} + \left(\frac{ds}{dt}\right)^2 \vec{T} \times \vec{T}'.$$

Because $|\vec{T}| = 1$, $\vec{T}$ and $\vec{T}'$ are orthogonal, so

$$|\vec{T} \times \vec{T}'| = |\vec{T}||\vec{T}'| \sin\frac{\pi}{2} = \underbrace{|\vec{T}|}_{=1}|\vec{T}'|, \text{ thus}$$

$$|\vec{r}' \times \vec{r}''| = \left(\frac{ds}{dt}\right)^2 |\vec{T}'| = |\vec{r}'|^2 |\vec{T}'|$$

Hence, by the first formula in the theorem:

$$k = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}.$$

$\square$

Exercise: recalculate the curvature of a circle of radius R using the formulas of the theorem.

<u>Ex</u>: Find the curvature of (t, t^2, t^3) at $(0,0,0)$.

$\vec{r}'(t) = (1, 2t, 3t^2)$, $\vec{r}''(t) = (0, 2, 6t)$, $|\vec{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (6t^2, -6t, 2)$$

need t.

$$\frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{1 + 4t^2 + 9t^4}^{\,3}}.$$ At $(0,0,0)$, $t = 0$, so

$k(0) = 2.$

Exercise: show that $\kappa(t) = 0$ for all t if and only if the curve is a straight line.

<u>Ex</u>: If a curve is on the xy-plane and given by $y = f(x)$,

$$\kappa(x) = \frac{f''(x)}{(1 + (f(x))^2)^{3/2}}.$$

To see this, write $\vec{r}(x) = (x, f(x), 0)$ and choose x as the parameter. $\vec{r}'(x) = (1, f'(x), 0)$, $\vec{r}''(x) = (0, f''(x), 0)$. $\vec{r}'(x) \times \vec{r}''(x) = (0, 0, f''(x))$. Then

$$\kappa(x) = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3} = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

Normal and binormal vectors

Since $\vec{T}' \cdot \vec{T} = 0$, $\vec{T}'$ and $\vec{T}$ are orthogonal. $\vec{T}'$ need not to be unit, but if $\kappa(t) \neq 0$, then $\vec{T}'(t) \neq \vec{0}$ (by $\kappa(t) = |\vec{T}'| / |\vec{r}'|$) so the unit vector

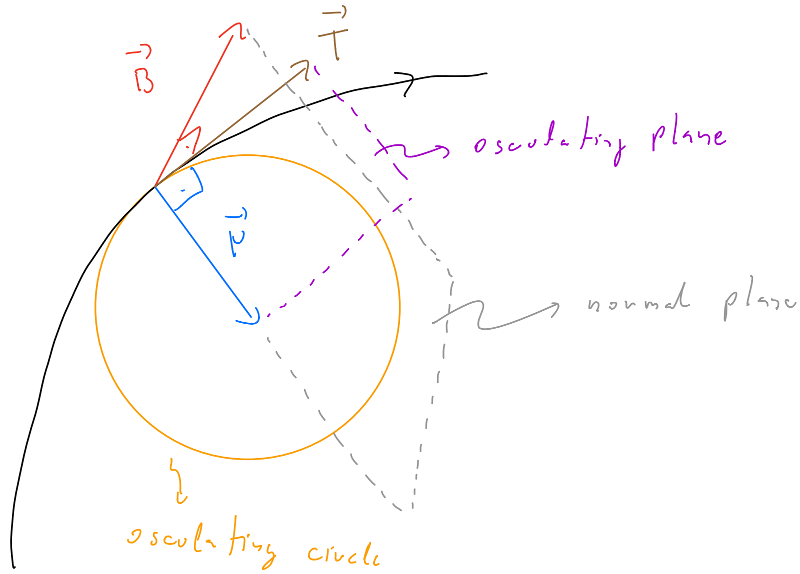
$$\vec{N}(t) := \frac{\vec{T}'(t)}{|\vec{T}'(t)|},$$

known as (principal) unit normal vector to the curve. The vector

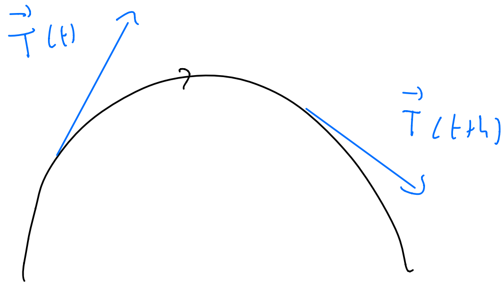
$$\vec{B}(t) := \vec{T}(t) \times \vec{N}(t),$$

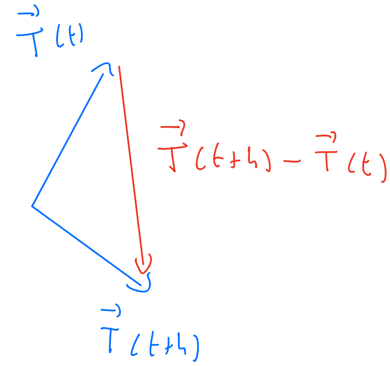
which is unit and orthogonal to both $\vec{T}$ and $\vec{N}$, is called the binormal vector.

At a given point P on a curve, the plane determined by $\vec{N}$ and $\vec{B}$ is called the normal plane at P, and that determined by $\vec{T}$ and $\vec{N}$ is called the osculating plane at P. The circle that lies on the osculating plane at P, has the same tangent at P, lies on the concave side of the curve (where $\vec{N}$ points) and has radius $1/k$ is called the osculating circle.



Why does $\vec{N}$ point to the concave part of the curve?





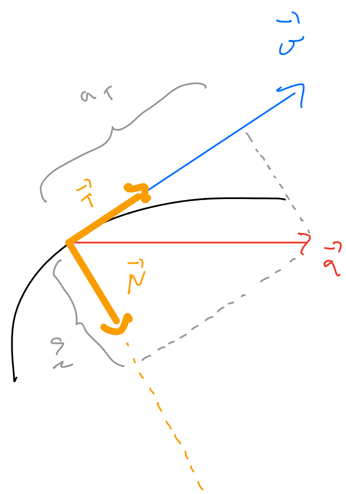
Motion in space: velocity and acceleration

If $\vec{r}(t)$ represents the position at time of a particle moving in space, then its velocity and acceleration at time t are given by, respectively,

$$\vec{v}(t) = \vec{r}\,'(t), \qquad \vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t).$$

$\vec{v}(t)$ is tangent to the particle's trajectory (the curve given by $\vec{r}(t)$) and $\vec{a}(t)$ points to the concave side of the trajectory (if it is not a straight line). Note that $\vec{v}(t)$ need not to be orthogonal. The particle's speed is $v(t) = |\vec{v}(t)|$.

One can decompose the acceleration into the directions tangent and perpendicular to the curve by projecting it onto $\vec{T}$ and $\vec{N}$, respectively. We write



$$\vec{a} = a_T \vec{T} + a_N \vec{N}.$$

Let us find a_T and a_N.

$$\vec{T} = \frac{\vec{v}}{v}, \quad \vec{v} = v\vec{T}$$

$$\vec{a} = \vec{v}\,' = v\vec{T}\,' + v'\vec{T}$$

$$\kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{|\vec{T}\,'|}{v}, \quad |\vec{T}\,'| = \kappa v,$$

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|}, \quad \vec{T}\,' = |\vec{T}\,'|\vec{N} = \kappa v \vec{N}.$$ Thus

$$\vec{a} = \vec{v}\,'\vec{T} + \kappa v^2 \vec{N}, \text{ so}$$

$$a_T = v'$$
$$a_N = \kappa v^2 \vec{N}$$

Observe that $a_N = 0$ (so $\vec{a}$ is tangent to the curve) iff $\kappa = 0$ or $v = 0$ (in both cases, the trajectory is a line).

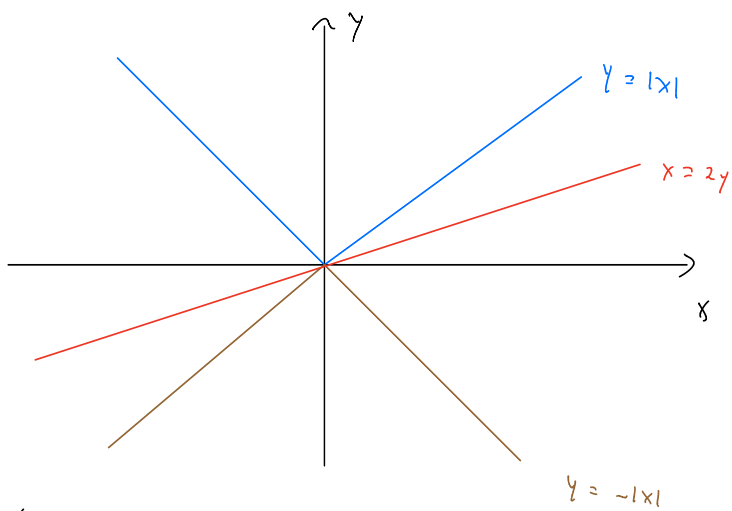
Functions of several variables

Def. A function of two variables is a rule that assigns to each order pair $(x,y) \in D \subset \mathbb{R}^2$ a unique real value $f(x,y)$. D is called the domain of f and $\{ f(x,y) \mid (x,y) \in D \}$ is called its range.

We often write $f = f(x,y)$ to mean "f is a function of two variables."

Ex: $f(x,y) = \dfrac{\sqrt{x^2 - y^2}}{x - 2y}$ is a function of two variables.

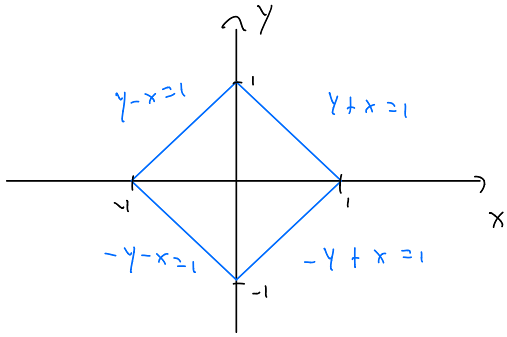
Its domain is determined by $x^2 - y^2 \geq 0$, so $|x| \geq |y|$, and $x - 2y \neq 0$, so $x \neq 2y$.



$D(f) = \{ (x,y) \in \mathbb{R}^2 \mid -|x| \leq y \leq |x| \} \cap \{ (x,y) \in \mathbb{R}^2 \mid x \neq 2y \}$.

Ex: Find the domain and range of $f(x,y) = \sqrt{1 - |x| - |y|}$

Note that to find the range first we need to find the domain. We need $1-|x|-|y| \geq 0$, $|x|+|y| \leq 1$.

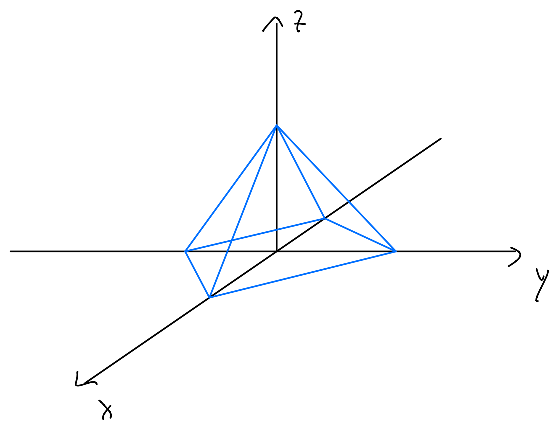


$f(0,0)=1$, $f(x,y)=0$ for $|x|+|y|=1$ and f decreases when (x,y) move from the origin toward the boundary.

So $R(f) = [0,1]$.

Def. The graph of a function $f=f(x,y)$ is the set of points $(x,y,z) \in \mathbb{R}^3$ such that $z=f(x,y)$ for $(x,y) \in D(f)$.

Ex: For the function in the previous example, the graph is sketched below



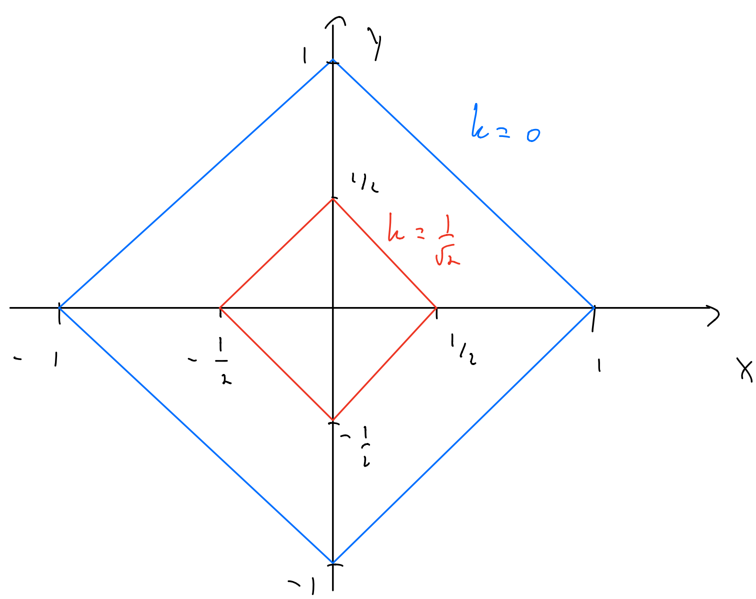
Ex: The graph of $f(x,y) = x^2+y^2$ is a paraboloid.

Level curves

Sometimes it is not easy to visualize the graph of a function $f = f(x,y)$. But we can still get an idea of its graph behavior by looking at level curves.

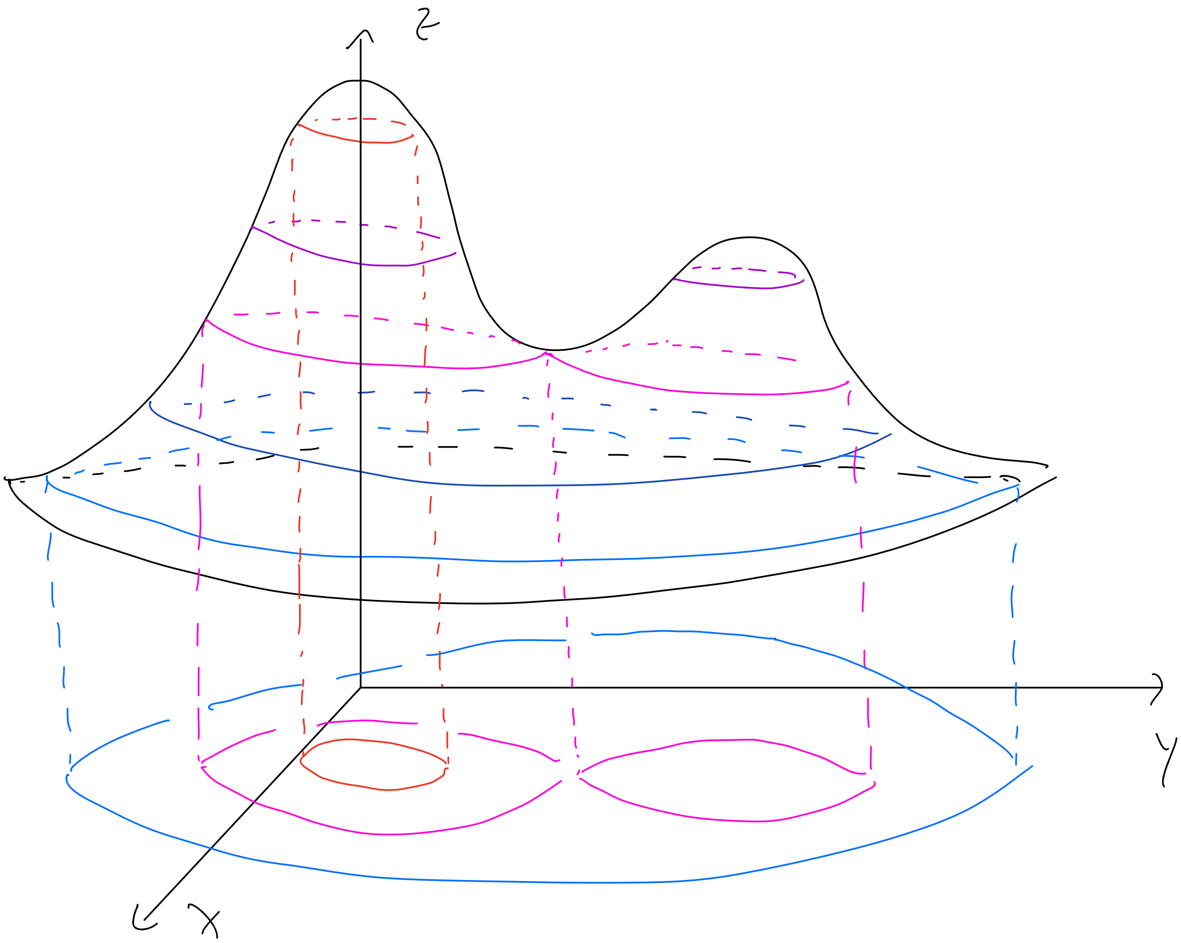
Def. The level curves of a function $f = f(x,y)$ are the curves with equation $f(x,y) = k$, where k is a constant.

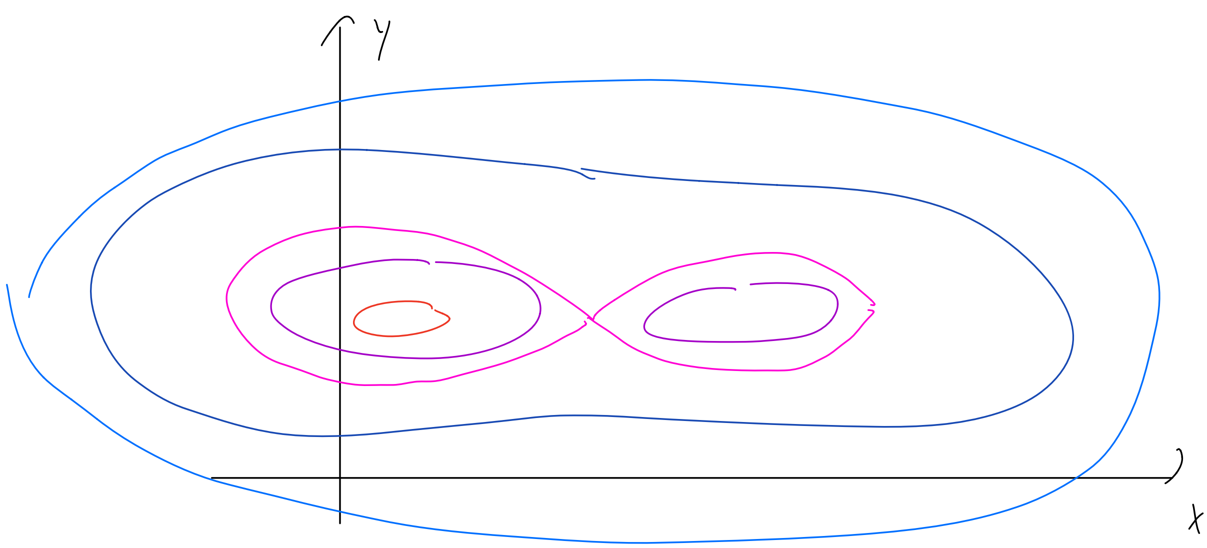
Ex: The level curves $f(x,y) = 0$ and $f(x,y) = \frac{1}{\sqrt{2}}$ are depicted below for $f(x,y) = \sqrt{1 - |x| - |y|}$.



Observe that the level curves $f(x,y) = k$ correspond to the intersections of graph(f) with the planes $z = k$.

Ex: Below a graph and some level curves are sketched. The level curves are color coded as indicated.





Functions of three or more variables

A function of three variables $f = f(x,y,z)$ is defined in the same way as a function of two variables, but now f takes a triple (x,y,z) as its argument.

Ex: The function $f(x,y,z) = \dfrac{1}{\sqrt{1-x^2-y^2-z^2}}$ has domain $x^2+y^2+z^2 < 1$. I.e., the domain of f is the set of all points inside the sphere of radius 1 centered at the origin in $\mathbb{R}^3$.

The sets $f(x,y,z) = k$, k constants, constitute surfaces in $\mathbb{R}^3$ called the level surfaces of f. For example, the level surfaces of $f(x,y,z) = x^2+2y^2+3z^2$ are ellipsoids, since for each number k, $x^2+2y^2+3z^2 = k$.

Similarly we can define functions of n variables $f(x_1, x_2, \dots, x_n)$.

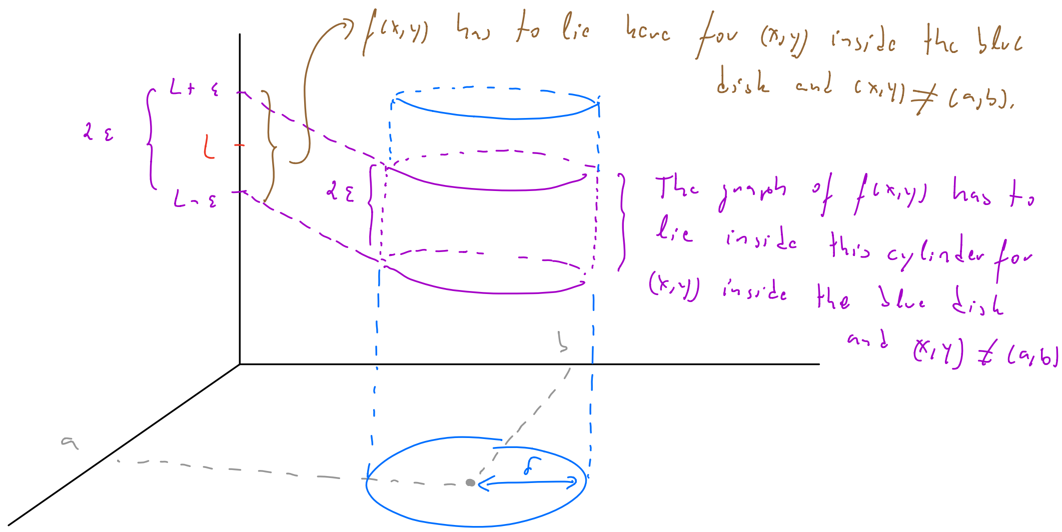
Limits and continuity

If f is a function of two variables with domain D, and $(a,b) \in D$, we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) equals L, and write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $(x,y) \in D$ and $0 < \underbrace{\sqrt{(x-a)^2 + (y-b)^2}}_{\text{distance from } (x,y) \text{ to } (a,b).} < \delta$ then

$$|f(x,y) - L| < \varepsilon.$$



The intuition is the same as in single variable calculus; $f(x,y)$ becomes arbitrarily close to L as (x,y) approaches (a,b).

Observe that as in the single variable case, the limit "doesn't care" about the value of $f(x,y)$ at $(x,y)=(a,b)$.

Ex: Let $f(x,y) = x^2$. Then $\lim_{(x,y)\to(1,0)} f(x,y) = 1$. Since $f(x,y)$ does not depend on y, we can think of it as a function of x only and then we know $x^2 \xrightarrow[x\to 1]{} 1$.

Ex: Let $f(x,y) = x^2+y^2$. What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

The limit is 0. To see this, note that

$$|f(x,y) - 0| = x^2+y^2.$$

Given $\varepsilon > 0$, if we take $\delta = \sqrt{\varepsilon}$

$$0 < \sqrt{x^2+y^2} < \delta \implies |f(x,y) - 0| < \varepsilon.$$

Ex: Let $f(x,y) = 2x^2+y^2$. What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

The limit is zero. To see this, given $\varepsilon > 0$, we want

$$|2x^2+y^2 - 0| = 2x^2+y^2 < \varepsilon \quad \text{if} \quad 0 < \sqrt{x^2+y^2} < \delta.$$

But $2x^2+y^2 \leq 2x^2+2y^2 \underset{\text{want}}{<} \varepsilon$, so if $\delta = \sqrt{\frac{\varepsilon}{2}}$

$$2x^2+y^2 \leq 2x^2+2y^2 = 2\underbrace{(x^2+y^2)}_{<\delta^2} < 2\delta^2 = \varepsilon.$$

A consequence of the definition is that for the limit to exist, we need to have $f(x,y)$ close to L regardless of how $(x,y) \to (a,b)$. Thus, we need to have $f(x,y) \to L$ for any path along which $(x,y) \to (a,b)$.

EX: what is $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$?

Write $f(x,y) = \dfrac{xy}{x^2+y^2}$.

Say (x,y) approaches $(0,0)$ along the x-axis, i.e., along $(x,0)$, $x \neq 0$.

Then $f(x,0) = \dfrac{x \cdot 0}{x^2+0^2} = 0$, so $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{(x,0)\to(0,0)} 0 = 0.$

Consider now the curve where $x=y$, and suppose $(x,y) \to (0,0)$ along this curve. Then $f(x,y) = \dfrac{x \cdot x}{x^2+x^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}.$$

Since we got two different values when approaching along different curves, the limit does not exist. This situation is the analogue of having the limits from the right and from the left to be different in single variable calculus.

Ex: Let $f(x,y) = \begin{cases} x^2+y^2, & (x,y) \neq (0,0), \\ 2, & (x,y) = (0,0). \end{cases}$

What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

Since the value at $(0,0)$ is not important for the limit, this is exactly as in the first example above and $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Continuity

We say that a function $f = f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(0,0)} f(x,y) = f(a,b)$. We say that f is continuous in region D if it is continuous for every $(a,b) \in D$. We say that f is continuous to mean that f is continuous for every (a,b) in its domain.

Thus, differently than what happens for limits, for continuity the value of f at (a,b) does matter.

Ex: Consider

$$f(x,y) = x^2 + y^2 \quad \text{and} \quad g(x,y) = \begin{cases} x^2+y^2, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0). \end{cases}$$

We have $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, $\lim_{(x,y)\to(0,0)} g(x,y) = 0$, so both limits agree, but $f(0,0) = 0$ and $g(0,0) = 2$. Thus, f is continuous at $(0,0)$ but g is not.

A polynomial (in the xy) variables is a sum of terms of the form $c\, x^m y^n$, c = constant, m,n integers ≥ 0. Polynomials are always continuous. A rational function is a quotient of two polynomials. Rational functions are continuous on their domain, i.e., for every (x,y) such that the denominator is not zero.

Ex: $x^2 + y^3$ is continuous, and $\dfrac{x^2+y^2}{x+y^3}$ is continuous for $x \neq y^3$.

Functions of three or more variables

The concepts of limit, continuity, polynomial and rational functions generalize in a straightforward manner to three or more variables.

Ex: $x^2+y^2+z^2$ is continuous and

$\dfrac{x^2+y^3}{x^2+y^2+2z^2-1}$ is continuous for all (x,y,z) except for those (x,y,z) on the ellipse $x^2+y^2+2z^2=1$.

Partial derivatives

We want to define derivatives of $f = f(x,y)$. Since there are two independent variables, we can imagine differentiating in the x-variable or the y-variable.

Def. If f is a function of two variables, its partial derivatives with respect to x and y, respectively, are the functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ given by

$$\frac{\partial f}{\partial x}(x,y) := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h},$$

$$\frac{\partial f}{\partial y}(x,y) := \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h},$$

provided the limits exist. Sometimes we denote $\frac{\partial f}{\partial x} = f_x = \partial_x f$, $\frac{\partial f}{\partial y} = f_y = \partial_y f$.

Therefore, $\frac{\partial f}{\partial x}$ is the derivative of $f(x,y)$ treated as a function of x only, i.e., holding y constant. Similarly,

$\frac{\partial f}{\partial y}$ is the derivative of $f(x,y)$ treated as a function of y only, i.e., holding x constant.

Ex: Let $f(x,y) = \cos x \sin y$. Then

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(\cos x \sin y) = \left(\frac{\partial}{\partial x}\cos x\right)\sin y = -\sin x \sin y$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(\cos x \sin y) = \cos x\left(\frac{\partial}{\partial y}\sin y\right) = \cos x \cos y.$$

Ex: Let $f(x,y) = e^{x^2+y^3}$. Then

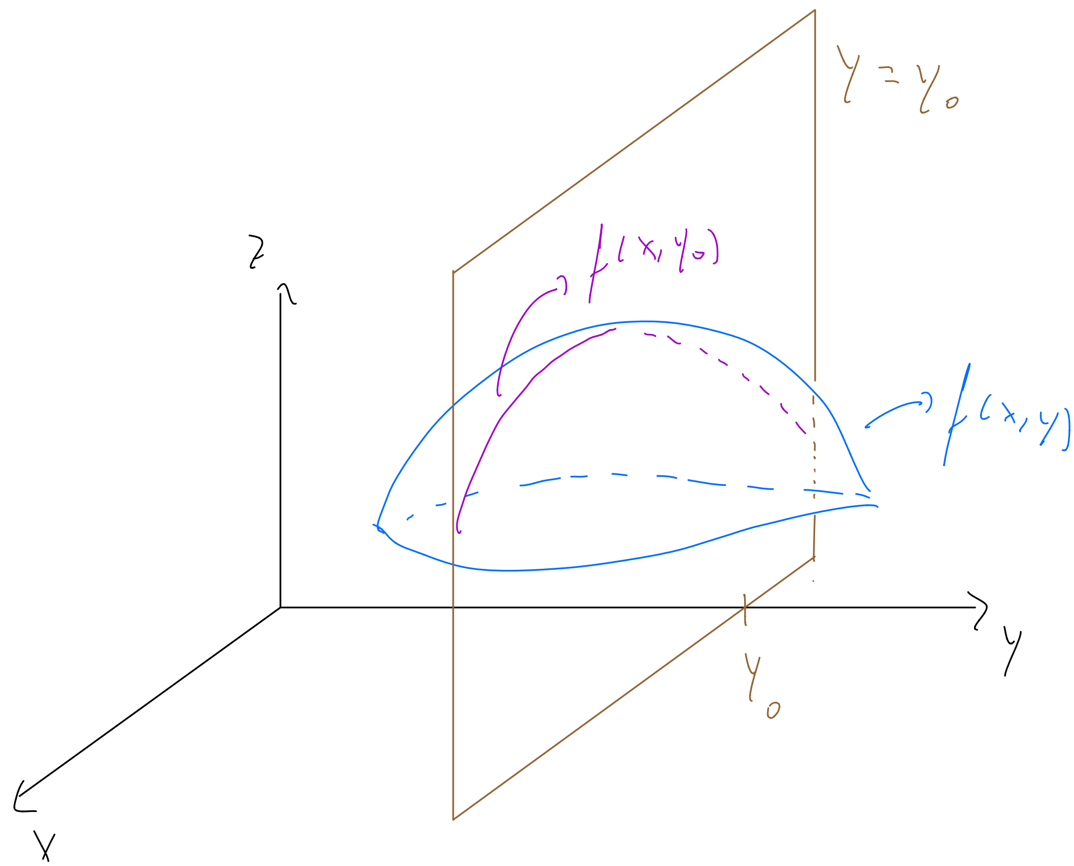
$$\frac{\partial f}{\partial x}(x,y) = 2x\, e^{x^2+y^3}$$

$$\frac{\partial f}{\partial y}(x,y) = 3y^2 e^{x^2+y^3}.$$

Ex: Find $f_y(0,0)$ for $f(x,y) = \cos x \sin y$.

$$f_y(0,0) = \cos 0 \cos 0 = 1.$$

When we compute $f_x(x_0,y_0)$, we can imagine taking the intersection of the graph of f with the plane $y = y_0$, and then taking the single variable derivative of the function $f(x, y_0)$ at $x = x_0$.



A similar interpretation holds for $f_y(x_0, y_0)$.

Higher derivatives

If f is a function of two variables, so are f_x and f_y. Thus we can take the partial derivatives of f_x and f_y, leading to the second order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx} = \partial_x(\partial_x f) = \partial^2_{xx} f$$

$$\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy} = \partial_y(\partial_x f) = \partial_{xy} f$$

$$\frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx} = \partial_x(\partial_y f) = \partial_y \partial_x f$$

$$\frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy} = \partial_y(\partial_y f) = \partial_y^2 f.$$

Similarly we define higher-order derivatives, e.g.,

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x \partial y}\right).$$

<u>Ex:</u> Let $f(x,y) = \frac{x^3}{y}$. Then

$$f_x(x,y) = \frac{3x^2}{y}, \quad f_y(x,y) = -\frac{x^3}{y^2}$$

$$f_{xx}(x,y) = \frac{6x}{y}, \quad f_{yy}(x,y) = \frac{2x^3}{y^3}$$

$$f_{xy}(x,y) = -\frac{3x^2}{y^2}, \quad f_{yx}(x,y) = -\frac{3x^2}{y^2}.$$

Note that $f_{xy} = f_{yx}$. This is <u>not</u> always true.

<u>Ex:</u> Let

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Then

$$\partial_y f(x,0) = \lim_{h\to 0} \frac{f(x,0+h) - f(x,0)}{h} = \lim_{h\to 0} \frac{\frac{xh(x^2-h^2)}{x^2+h^2}}{h} = x,$$

$$\partial_x f(0,y) = \lim_{h\to 0} \frac{f(0+h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{\frac{hy(h^2-y^2)}{h^2+y^2}}{h} = -y.$$

Then $\partial_x \partial_y f(0,0) = 1$, $\partial_y \partial_x f(0,0) = -1$.

So, when is the case that $\partial_{xy} f = \partial_{yx} f$?

<u>Clairaut's theorem.</u> If $f = f(x,y)$ is defined on a disk D containing the point (a,b), and the functions f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

In this case, we say that the <u>partial derivatives commute</u>.

Similarly, continuity of higher-order derivatives gives their commutation, e.g., $\partial_{xyy} f = \partial_{yyx} f = \partial_{yxy} f$.

Partial derivatives of functions of several variables

The above concepts generalize to functions of more than two variables.

Ex: Let $f(x,y,z) = x^2 y z^3 + \cos(x z^2)$. Then

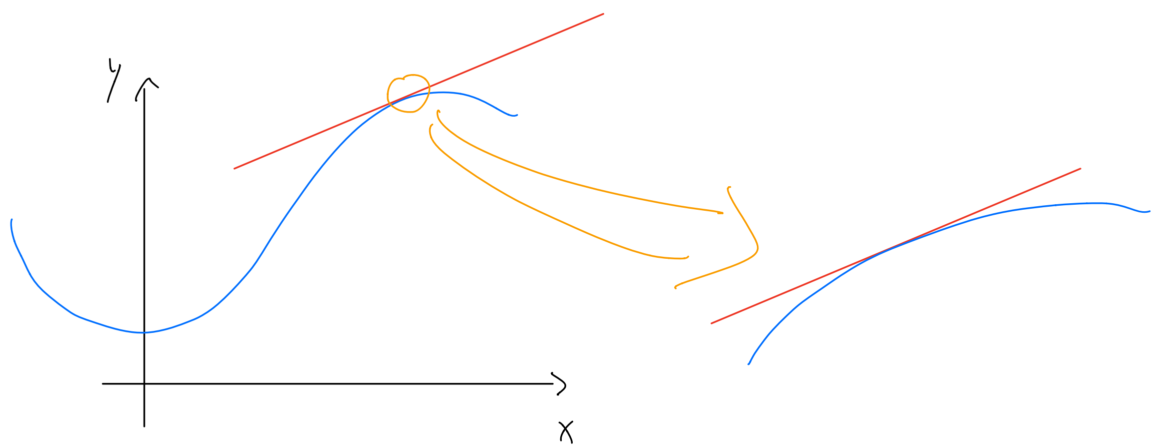
$$f_x(x,y,z) = 2xyz^3 - \sin(xz^2)$$

$$f_y(x,y,z) = x^2 z^3$$

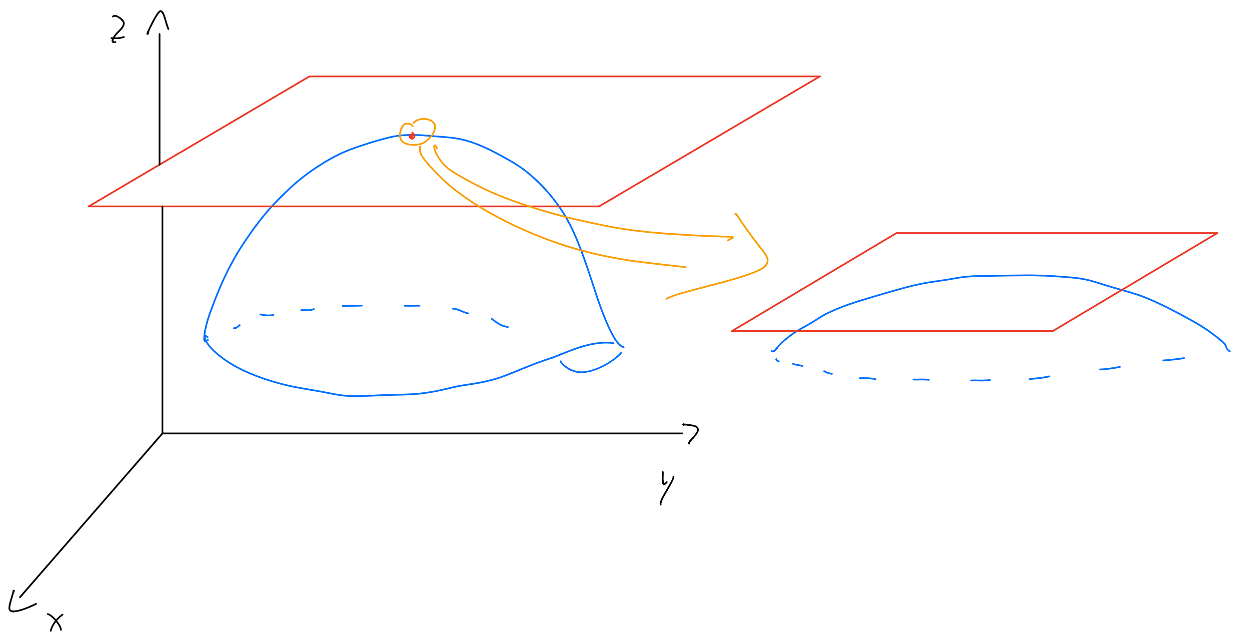
$$f_z(x,y,z) = 3x^2 y z^2 - 2xz \sin(xz^2).$$

Tangent planes and linear approximations

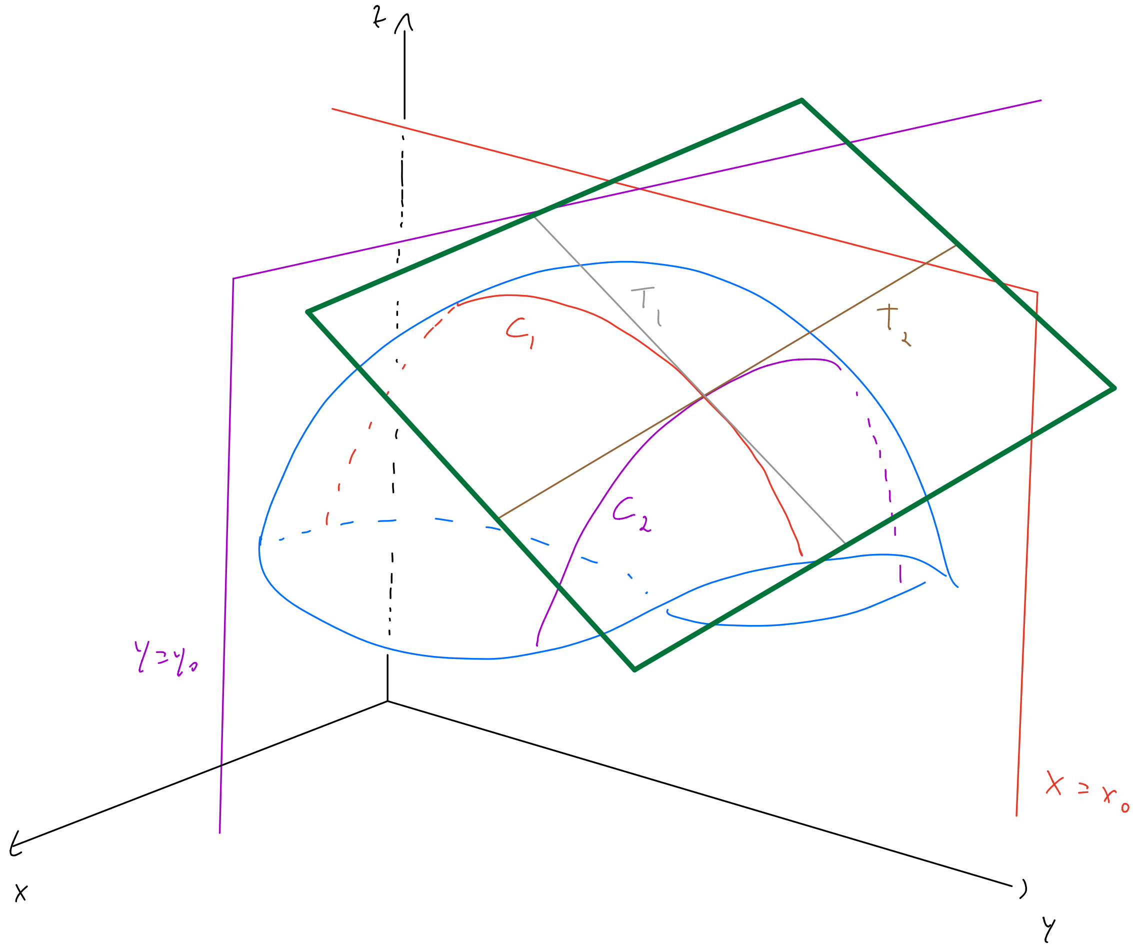
In single variable calculus, when we zoom in toward a point in a graph of a differentiable function, the graph becomes very close to the tangent line at that point.



We say that the graph is approximated by the tangent line. Similarly, for a function $f = f(x,y)$, its graph at a point will be approximated by a plane, the tangent plane to the graph at that point.



Consider a surface S in $\mathbb{R}^3$ given by $z = f(x,y)$, where f has continuous first partial derivatives. Let (x_0, y_0, z_0) be a point in S, so $z_0 = f(x_0, y_0)$. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $x = x_0$ and $y = y_0$ with S, respectively. Note that (x_0, y_0, z_0) belongs to both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2, respectively, at (x_0, y_0, z_0). The <u>tangent plane</u> to S at (x_0, y_0, z_0) is defined as the plane that contains both T_1 and T_2.



How do we find the equation of the tangent plane, given that we know the equation of S (i.e., $z = f(x,y)$, and f is known)? Since the plane passes through (x_0, y_0, z_0), we know that it has the form:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

We need to determine A, B, and C. Equivalently, dividing by C ($C \neq 0$) and writing

$$a = -\frac{A}{C}, \qquad b = -\frac{B}{C},$$

we have

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

Thus, it suffices to determine a and b. Setting $y = y_0$ we have

$$z - z_0 = a(x - x_0),$$

which is the equation of a line in the zx-plane with slope a. By construction, this line belongs both to the tangent plane and to the plane $y = y_0$, i.e., it is the line T_2. Now, T_2 is the tangent line to the curve C_2 at (x_0, y_0, z_0). Since C_2 is obtained by intersecting $z = f(x, y)$ with $y = y_0$, thus C_2 has equation

$$z = f(x, y_0),$$

which gives z as a function of x, i.e., a curve in the xz-plane (the curve C_2 projected on the xz-plane). Therefore, the slope of the line tangent to C_2 at (x_0, y_0, z_0) is $\frac{\partial f}{\partial x}(x_0, y_0)$, i.e.,

$$a = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = \frac{\partial f}{\partial x}(x_0, y_0).$$

Similarly, taking $x = x_0$ we have the curve C_1

$$z = f(x_0, y)$$

and we obtain

$$b = \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = \frac{\partial f}{\partial y}(x_0, y_0).$$

In sum, the equation of the tangent plane to $z = f(x,y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Since $z_0 = f(x_0, y_0)$, we can also write

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

<u>Ex</u>: Find the equation of the tangent plane to $z = x^2 + y^2$ at $(1, 1, 2)$.

First, note that $(1,1,2)$ indeed belongs to $z = x^2 + y^2$. Computing,

$$z_x = 2x, \quad z_y = 2y, \quad \text{so}$$

$$a = z_x(1,1) = 2, \quad b = z_y(1,1) = 2,$$

thus

$$z - 2 = 2(x - 1) + 2(y - 1).$$

<u>Linear approximations</u>

Given $f(x,y)$, the tangent plane to $z = f(x,y)$ at (x_0, y_0, z_0), view as a function of x and y, is called the <u>linearization of f</u> at (x_0, y_0), denoted

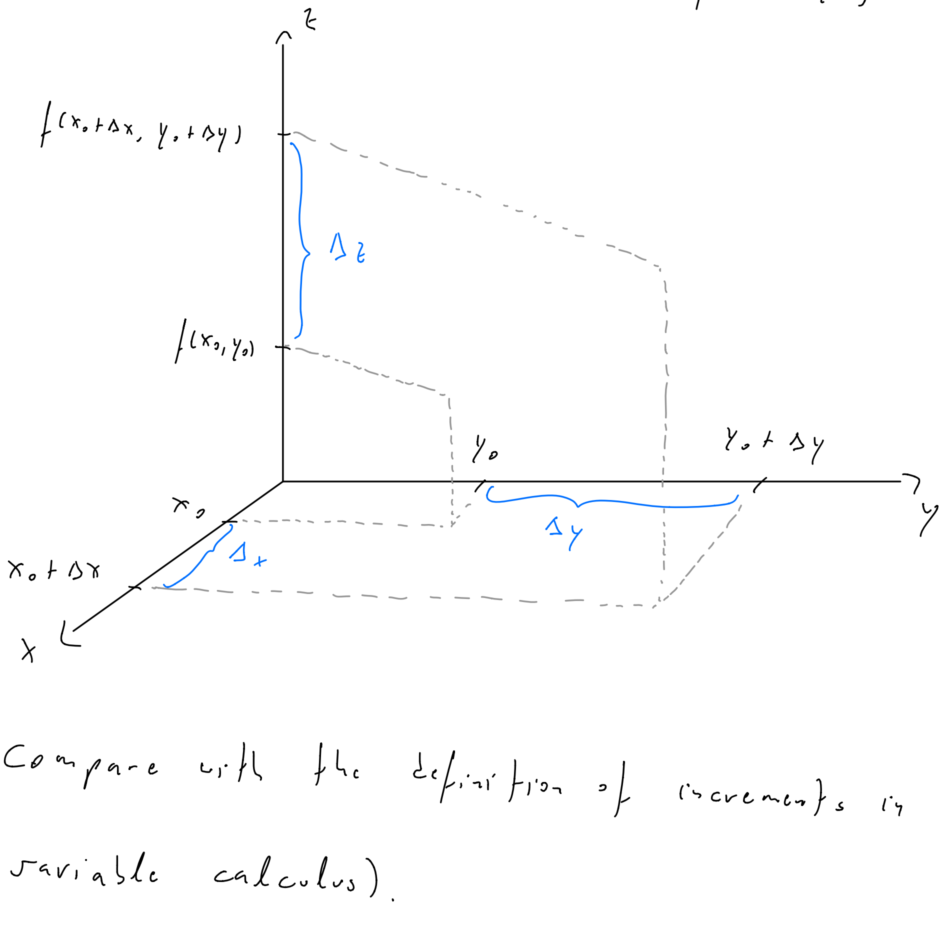
$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

We can think of the linearization as an approximation of f near (x_0, y_0):

$$f(x,y) \approx L(x,y) \quad \text{for} \quad (x,y) \approx (x_0,y_0).$$

We would like to make this approximation statement more precise. For given numbers Δx and Δy, we define the increment of z when (x_0, y_0) changes to $(x_0+\Delta x, y_0+\Delta x)$ as

$$\Delta z := f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$



(Compare with the definition of increments in single variable calculus).

Def. A function $f = f(x,y)$ is differentiable at (x_0, y_0) if, setting $z = f(x,y)$, Δz can be written as

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. We say that f is differentiable if it is differentiable at every (x_0,y_0) in its domain.

Writing $x = x_0 + \Delta x$, $y = y_0 + \Delta x$, so that $\Delta z = f(x,y) - f(x_0,y_0)$, we have

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y}_{= L(x,y)} + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$f(x,y) = L(x,y) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Therefore, L is a good approximation to f (the error $\varepsilon_1, \varepsilon_2$ goes to zero as $(\Delta x, \Delta y) \to (0,0)$) if f is differentiable at (x_0,y_0).

Ex: Show that $f(x,y) = x^2 + y^2$ is differentiable at $(1,1)$

$$f_x(x,y) = 2x, \quad f_y(x,y) = 2y$$

$$f_x(1,1) = 2, \quad f_y(1,1) = 2$$

$$f_x(1,1)\Delta x + f_y(1,1)\Delta y = 2\Delta x + 2\Delta y$$

$$\begin{aligned}
\Delta z &= f(1+\Delta x, 1+\Delta x) - f(1,1) \\
&= (1+\Delta x)^2 + (1+\Delta y)^2 - 2 \\
&= 1 + (\Delta x)^2 + 2\Delta x + 1 + (\Delta y)^2 + 2\Delta y - 2 \\
&= \underbrace{2\Delta x + 2\Delta y}_{= f_x(1,1)\Delta x + f_y(1,1)\Delta y} + (\Delta x)^2 + (\Delta y)^2 \\
&= f_x(1,1)\Delta x + f_y(1,1)\Delta y + \underbrace{\Delta x}_{= \varepsilon_1} \cdot \Delta x + \underbrace{\Delta y}_{\varepsilon_2} \cdot \Delta y.
\end{aligned}$$

We see that $\varepsilon_1, \varepsilon_2 \longrightarrow 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Ex: Show that

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = 0 \end{cases}$$

is not differentiable at $(0,0)$.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h \cdot 0}{h^2+0^2} \cdot \frac{1}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 \cdot h}{0^2+h^2} \cdot \frac{1}{h} = 0$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$$

$$= \frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2} = \underbrace{0 \Delta x + 0 \Delta y}_{f_x(0,0)\Delta x + f_y(0,0)\Delta y} + \frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2}$$

Now we ask if it is possible to write

$$\frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2} = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. If this is possible, then in particular

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\varepsilon_1 \Delta x + \varepsilon_2 \Delta y \right) = 0.$$

But we saw that the limit

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$$

does not exist, so

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x \, \Delta y}{(\Delta x)^2 + (\Delta y)^2}$$

does not exist, thus we cannot have

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x \, \Delta y}{(\Delta x)^2 + (\Delta y)^2} = 0$$

and f is not differentiable at $(0,0)$.

The following theorem is useful to determine if f is differentiable at (x_0, y_0).

Theo. If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Ex: Consider again $f(x,y) = x^2 + y^2$. $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$ are polynomials, hence continuous for any (x,y). Thus f is differentiable.

An important fact about differentiability is the following: if $f = f(x,y)$ is differentiable at (x_0, y_0) then it is continuous at (x_0, y_0). To see this, consider the difference $f(x,y) - f(x_0, y_0)$ and write $x = x_0 + \underbrace{(x - x_0)}_{= \Delta x} = x_0 + \Delta x$ and

$y = y_0 + \underbrace{(y - y_0)}_{= \Delta y} = y_0 + \Delta y$. Then

$$f(x,y) - f(x_0,y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0,y_0)$$
$$= \Delta f.$$

Since f is differentiable at (x_0,y_0)

$$f(x,y) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y$$
$$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Since the RHS goes to zero when $(\Delta x, \Delta y) \to (0,0)$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$|(\Delta x, \Delta y)| = \sqrt{(\Delta x)^2 + (\Delta y)^2} < \delta$$

then

$$\left| \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \right| < \varepsilon.$$

Thus, if (x,y) satisfy

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2} < \delta$$

we obtain $|f(x,y)-f(x_0,y_0)|<\varepsilon$, showing continuity.

Remark. The existence of only the partial derivatives does not guarantee continuity, e.g.,

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$ satisfies

$f_x(0,0)=0$, $f_y(0,0)=0$ but f is not continuous at $(0,0)$.

Differentials

Recall that in single variable calculus we define the differential of $y=f(x)$ as

$$dy=f'(x)\,dx.$$

Similarly, for $z=f(x,y)$ we define the differential

$$dz := \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Intuitively, we think of dz, dx, and dy as approximations of Δz, Δx, and Δy when these quantities are very small.

<u>Ex:</u> Let $z = f(x,y) = x^2 + 3xy - y^2$.

- Find dz

- Find Δz when x changes from 2 to 2.05 and y changes from 3 to 2.96, and compare with dz.

Since $f_x(x,y) = 2x + 3y$, $f_y(x,y) = 3x - 2y$,

$$dz = (2x+3y)dx + (3x-2y)dy.$$

Next, compute

$$\Delta z = f(2.05, 2.96) - f(2,3)$$

$$= (2.05)^2 - 3\cdot 2.05\cdot 2.96 - (2.96)^2 - (2^2 + 3\cdot 2\cdot 3 - 3^2)$$

$$= 0.6449.$$

To compare, in the formula for dz, plug

$x = 2,\ y = 3,\ dx \approx \Delta x = 0.05,\ dy \approx \Delta y = -0.04$:

$$dz \approx (2 \cdot 2 + 3 \cdot 3)(0.05) + (3 \cdot 2 - 2 \cdot 3) \cdot (-0.04)$$

$$= 13 \cdot 0.05 = 0.65.$$

The point is that $\Delta z \approx dz$, but dz is easier to compute.

Functions of three or more variables

All the previous concepts generalize to functions of three or more variables. For example,

$$f(x,y,z) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0)$$
$$+ f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(y - z_0).$$

The chain rule

Recall the chain rule for functions of one variable: if $y = f(x)$, and $x = g(t)$, and f and g are differentiable, then the composition $y = f(g(t))$ is differentiable and

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt},$$

or equivalently

$$(f(g(t)))' = f'(g(t))\,g'(t).$$

We will generalize this for functions of several variables. Considering first functions of at most two variables, there are different cases to consider:

- $z = f(x,y)$, $x = g(t)$, $y = h(t)$,

$$z(t) = f(x(t), y(t)) = f(g(t), h(t))$$

- $z = f(t)$, $t = g(x,y)$

$$z(x,y) = f(t(x,y)) = f(g(x,y))$$

- $z = f(x,y)$, $x = g(t,s)$, $y = h(t,s)$

$$z(t,s) = f(x(t,s), y(t,s)) = f(g(t,s), h(t,s)).$$

The chain rule, case I. Suppose that $z = f(x,y)$ is a differentiable function. Let $x = g(t)$ and $y = h(t)$ be differentiable. Then, the composition

$$z(t) = f(x(t), y(t))$$

is differentiable (when defined) and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

which we can also write as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

To see why the result is true, consider

$$\Delta z = z(t+\Delta t) - z(t)$$

$$= f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t)).$$

Since x and y are differentiable

$$x(t+\Delta t) = x(t) + \frac{dx}{dt}\Delta t + \varepsilon_{(x)}\Delta t$$

$$y(t+\Delta t) = y(t) + \frac{dy}{dt}\Delta t + \varepsilon_{(y)}\Delta t$$

where $\varepsilon_{(x)}, \varepsilon_{(y)} \to 0$ as $\Delta t \to 0$. Thus, since f is differentiable

$$\Delta z = f\Big(x(t) + \underbrace{\frac{dx}{dt}\Delta t + \varepsilon_{(x)}\Delta t}_{\Delta x},\ y(t) + \underbrace{\frac{dy}{dt}\Delta t + \varepsilon_{(y)}\Delta t}_{\Delta y} \Big)$$

$$= \frac{\partial f}{\partial x}(x(t), y(t))\,\Delta x + \frac{\partial f}{\partial y}(x(t), y(t))\,\Delta y$$

$$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta x.$$

Divide by Δt and take the limit $\Delta t \to 0$. Note that

$$\frac{\Delta x}{\Delta t} = \frac{dx}{dt} + \varepsilon_{(x)} \longrightarrow \frac{dx}{dt} \quad \text{as} \quad \Delta t \to 0,$$

$$\frac{\Delta y}{\Delta t} = \frac{dy}{dt} + \varepsilon_{(y)} \longrightarrow \frac{dy}{dt} \quad \text{as} \quad \Delta t \to 0,$$

$$\Delta x \to 0 \quad \text{as} \quad \Delta t \to 0, \quad \Delta y \to 0 \quad \text{as} \quad \Delta t \to 0.$$

Thus

$$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

<u>Ex</u>: Let $f(x,y) = \cos(x^2 y)$, $x(t) = e^{2t}$, $y(t) = t^3$.

Find $\frac{df}{dt}$ (i) using the chain rule, (ii) by direct substitution

(i) $\frac{\partial f}{\partial x} = -2xy\sin(x^2 y)$, $\frac{\partial f}{\partial y} = -x^2 \sin(x^2 y)$

$\frac{dx}{dt} = 2e^{2t}$, $\frac{dy}{dt} = 3t^2$

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= -2xy\sin(x^2 y)\cdot 2e^{2t} - x^2\sin(x^2 y)\,3t^2 \\
&= -4e^{4t}t^3\sin(e^{4t}t^3) - 3e^{4t}t^2\sin(e^{4t}t^3) \\
&= -(4t+3)\,e^{4t}t^2\sin(e^{4t}t^3).
\end{aligned}$$

(ii) $f(t) = \cos(e^{4t}t^3)$

$$\begin{aligned}
f'(t) &= -\sin(e^{4t}t^3)\,(4e^{4t}t^3 + 3e^{4t}t^2) \\
&= -(4t+3)\,e^{4t}t^2\sin(e^{4t}t^3).
\end{aligned}$$

This example might cause the impression that one can always replace x and y, as in (ii), and compute the derivative as in single variable calculus, dispensing with the chain rule in several variables. But this only works when x and y depend only on one variable. Otherwise we have to use partial derivatives.

Chain rule, case II. Suppose that $z = f(x,y)$ is differentiable. Let $x = g(t,s)$, $y = h(t,s)$ be differentiable. Then the composition

$$z(x(t,s), y(t,s))$$

is differentiable (when defined) and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

The idea is that now we apply case I separately for each variable t and s, and the derivatives are always partial derivatives.

<u>Ex</u>: Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ if $z(x,y) = e^{x+y^2}$, $x(t,s) = s+t$, $y(t,s) = \frac{s}{t}$ (i) using the chain rule, (ii) by direct substitution.

(i) $\frac{\partial z}{\partial x} = e^{x+y^2}$, $\frac{\partial z}{\partial y} = 2y\, e^{x+y^2}$

$\frac{\partial x}{\partial t} = 1$, $\frac{\partial y}{\partial t} = -\frac{s}{t^2}$, $\frac{\partial x}{\partial s} = 1$, $\frac{\partial y}{\partial s} = \frac{1}{t}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^{x+y^2}\cdot 1 + 2y\, e^{x+y^2}\left(-\frac{s}{t^2}\right)$$

$$= \left(1 - 2\frac{s^2}{t^3}\right) e^{s+t+\frac{s^2}{t^2}}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^{x+y^2}\cdot 1 + 2y\, e^{x+y^2}\,\frac{1}{t}$$

$$= \left(1 + 2\frac{s}{t^2}\right) e^{s+t+\frac{s^2}{t^2}}.$$

(ii) $z(s,t) = e^{s+t+\frac{s^2}{t^2}}$

$$\frac{\partial z}{\partial t} = e^{s+t+\frac{s^2}{t^2}} \cdot \frac{\partial}{\partial t}\left(s+t+\frac{s^2}{t^2}\right)$$

$$= \left(1 - \frac{2s^2}{t^3}\right) e^{s+t+\frac{s^2}{t^2}}.$$

$$\frac{\partial z}{\partial s} = e^{s+t+\frac{s^2}{t^2}} \frac{\partial}{\partial s}\left(s+t+\frac{s^2}{t^2}\right)$$

$$= \left(1 + \frac{2s}{t^2}\right) e^{s+t+\frac{s^2}{t^2}}.$$

In case II, we have s and t as independent variables, z as the dependent variable, and x and y as intermediate variables.

$$\begin{tikzcd}[column sep=small]
 & & & z \arrow[dll, dash, "\frac{\partial z}{\partial x}"'] \arrow[drr, dash, "\frac{\partial z}{\partial y}"] & & & \\
 & x \arrow[dl, dash, "\frac{\partial x}{\partial t}"'] \arrow[dr, dash, "\frac{\partial x}{\partial s}"] & & & & y \arrow[dl, dash, "\frac{\partial y}{\partial t}"'] \arrow[dr, dash, "\frac{\partial y}{\partial s}"] & \\
t & & s & & t & & s
\end{tikzcd}$$

<u>Chain rule, case III</u>. Suppose $z = f(x)$ is differentiable. Let $x = g(t, s)$ be differentiable. Then the composition

$$z(x(t,s))$$

is differentiable (when defined) and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}.$$

This is simply case II in the particular case when $z(x,y) = z(x)$, i.e., z does not depend on y (so $\frac{\partial z}{\partial y} = 0$).

<u>Chain rule, general case</u>. Suppose that z is a differentiable function of n variables $x_1, \dots, x_n$,

$$z = z(x_1, x_2, \dots, x_n),$$

and each x_j is a differentiable function of m variables $t_1, \dots, t_m$,

$$x_j = x_j(t_1, \dots, t_m).$$

Thus, the composition

$$z(x_1(t_1,\dots,t_m), x_2(t_1,\dots,t_m), \dots, x_n(t_1,\dots,t_m))$$

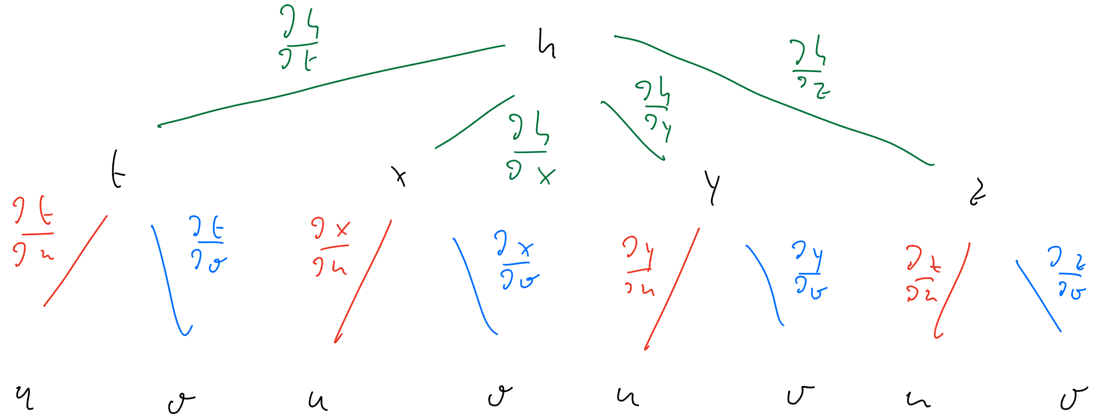
is differentiable (when defined) and

$$\frac{\partial z}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} \frac{\partial x_j}{\partial t_i}, \quad i = 1, \dots, m.$$

<u>EX</u>: Write the chain rule for $h = f(t, x, y, z)$ and t, x, y, z are functions of (u, v).

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial h}{\partial v} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$



Implicit differentiation

Assume that we have an equation $F(x,y)=0$ defining y implicitly as a function of x, $y=y(x)$, and that F and y are differentiable. Then, by the chain rule

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

So

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

provided $\frac{\partial F}{\partial y} \neq 0$.

Similarly, assume that $z=f(x,y)$ is defined implicitly by $F(x,y,z)=0$. The chain rule gives

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

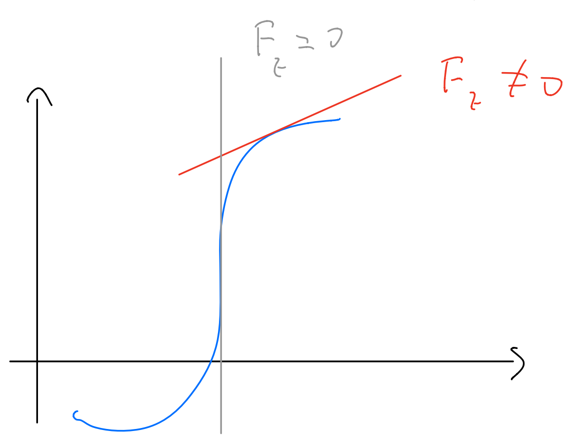
Thus

$$\frac{\partial z}{\partial x} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}. \quad \text{Similarly: } \frac{\partial z}{\partial y} = - \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

It remains to know when an equation as $F(x,y,z)=0$ defines z implicitly as a function of (x,y). For this we have:

Implicit function theorem: Assume that F is defined on ball B containing the point (a,b,c), $F(a,b,c)=0$, $F_z(a,b,c) \neq 0$, and F_x, F_y, and F_z are continuous in B. Then, $F(x,y,z)=0$ defines z implicitly in terms of (x,y) near (a,b,c), and

$$z_x = -\frac{F_x}{F_z}, \quad z_y = -\frac{F_y}{F_z}.$$

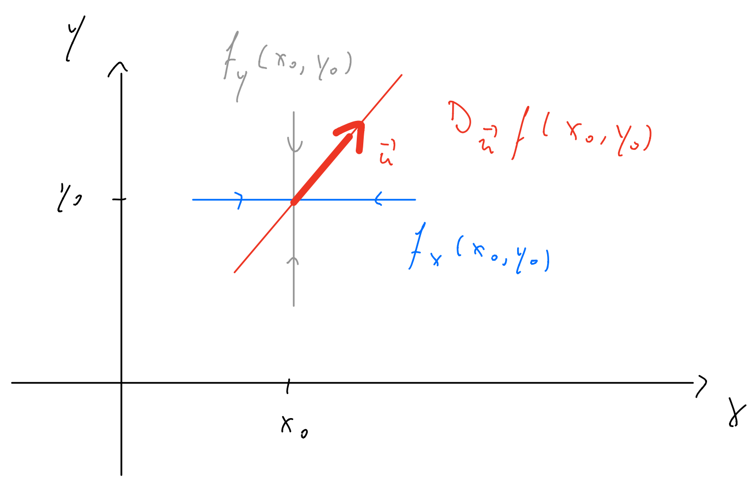


Directional derivatives and the gradient vector

The partial derivatives $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ are obtained by approaching (x_0,y_0) along lines parallel to the x- and y-axis, respectively. Given a non-zero vector $\vec{u} = (a,b)$, we can approach (x_0,y_0) along the line spanned by $\vec{u}$. We define the directional derivative of f in the direction of $\vec{u} = (a,b)$ at (x_0,y_0) as

$$D_{\vec{u}} f(x_0,y_0) := \lim \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

provided the limit exists.



Note that

$$D_{\vec{i}} f(x_0, y_0) = f(x_0, y_0) \quad \text{and} \quad D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)$$

When $D_{\vec{u}} f$, $\vec{u} = (a,b)$, f_x, and f_y all exist they are related by

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b.$$

Observe that the RHS is the dot product between the vectors $(f_x(x,y), f_y(x,y))$ and (a,b). This motivates defining the gradient vector of f (or simply the gradient of f) as

$$\operatorname{grad} f(x,y) = \vec{\nabla} f(x,y) = \nabla f(x,y) = (f_x(x,y), f_y(x,y)),$$

so that

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

Ex: Find ∇f if $f(x,y) = x^3 e^{xy}$.

Find $D_{\vec{u}} f$ if $\vec{u} = (2,-1)$.

Compute

$$f_x(x,y) = 3x^2 e^{xy} + x^3 y e^{xy} = (3x^2 + x^3 y) e^{xy}$$

$$f_y(x,y) = x^4 e^{xy}$$

Thus

$$\nabla f(x,y) = (3x^2 + x^3 y) e^{xy}\,\vec{i} + x^4 e^{xy}\,\vec{j}$$

Next,

$$\begin{aligned} D_{\vec{u}} f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\ &= \left((3x^2 + x^3 y) e^{xy},\ x^4 e^{xy}\right) \cdot (2,-1) \\ &= (6x^2 + 2x^3 y) e^{xy} - x^4 e^{xy} \\ &= (6x^2 + 2x^3 y - x^4) e^{xy}. \end{aligned}$$

The same way f_x and f_y measure the rate of change of f in the x and y directions, $D_{\vec{u}} f$ measures the rate of change of f in the $\vec{u}$-direction. We can ask in which direction (i.e., for which $\vec{u}$) is the rate of change maximum. Note that

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta$$

This depends on $|\vec{u}|$, but if we want to identify the direction that maximizes the rate of change of f, should take $|\vec{u}| = 1$, so

$$D_{\vec{u}} f = |\nabla f| \cos\theta .$$

This is maximized when $\theta = 0$, i.e., when ∇f and $\vec{u}$ are parallel.

Ex: In which direction does $f(x,y) = x^3 e^{xy}$ have maximum rate of change.

Since $\nabla f(x,y) = (3x^2 + x^3 y) e^{xy} \vec{i} + x^4 e^{xy} \vec{j}$,

we need to find $\vec{u}$ that is unit and parallel to ∇f.

thus

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{(3x^2 + x^3 y) e^{xy} \vec{i} + x^4 e^{xy} \vec{j}}{\sqrt{(3x^2 + x^3 y)^2 e^{2xy} + x^8 e^{2xy}}}$$

$$= \frac{(3x^2 + x^3 y) \vec{i} + x^4 \vec{j}}{\sqrt{(3x^2 + x^3 y)^2 + x^8}}.$$

Tangent planes to level surfaces

The previous notions generalize to $f = f(x,y,z)$ and functions of more variables.

$$\nabla f = (f_x, f_y, f_z).$$

$$D_u f = \nabla f \cdot \vec{u}.$$

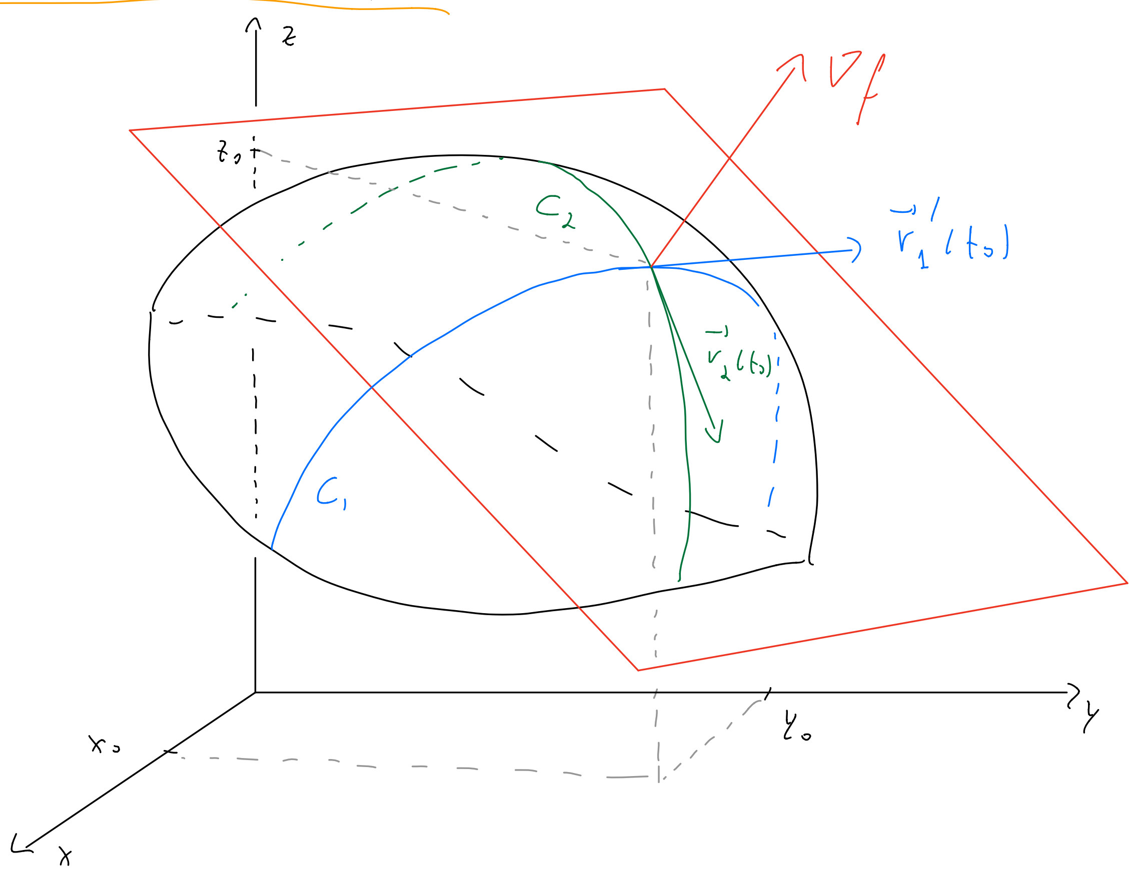
Consider a level surface of f, $f(x,y,z) = k$, and

Let $\vec{r}(t) = (x(t), y(t), z(t))$ be a curve on this surface, so $f(x(t), y(t), z(t)) = k$. Taking $\frac{d}{dt}$ and using the chain rule:

$$\frac{d}{dt} f = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$= \nabla f \cdot \vec{r}\,' = 0.$$

Thus, for any $(x_0, y_0, z_0) = \vec{r}(t_0)$, the tangent vector to $\vec{r}(t)$ at t_0 is orthogonal to $\nabla f(x_0, y_0, z_0)$. Hence $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent plane to the level surface $f(x, y, z) = k$ at (x_0, y_0, z_0).



Therefore, the equation of the plane tangent to $f(x, y, z) = k$ at (x_0, y_0, z_0) is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0) = 0.$$

The normal line to $f(x, y, z)$ at (x_0, y_0, z) is the line perpendicular to this plane, which can be written as

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}.$$

In the particular case when the surface is given by a graph $z = f(x, y)$, we can write $F(x, y, z) = f(x, y) - z$ and apply the above to F, so

$$\nabla F(x_0, y_0, z_0) = (f_x(x_0, y_0), f_y(x_0, y_0), -1).$$

Ex: Find an equation for the plane tangent to the sphere $x^2 + y^2 + z^2 = 1$ at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, and the corresponding normal line.

The sphere is the level surface $f(x,y,z) = 1$ of $f(x,y,z) = x^2 + y^2 + z^2$. Thus

$$\nabla f(x,y,z) = (2x, 2y, 2z),$$

$$\nabla f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) = \left(1, 1, \frac{1}{\sqrt{2}}\right)$$ and the plane is

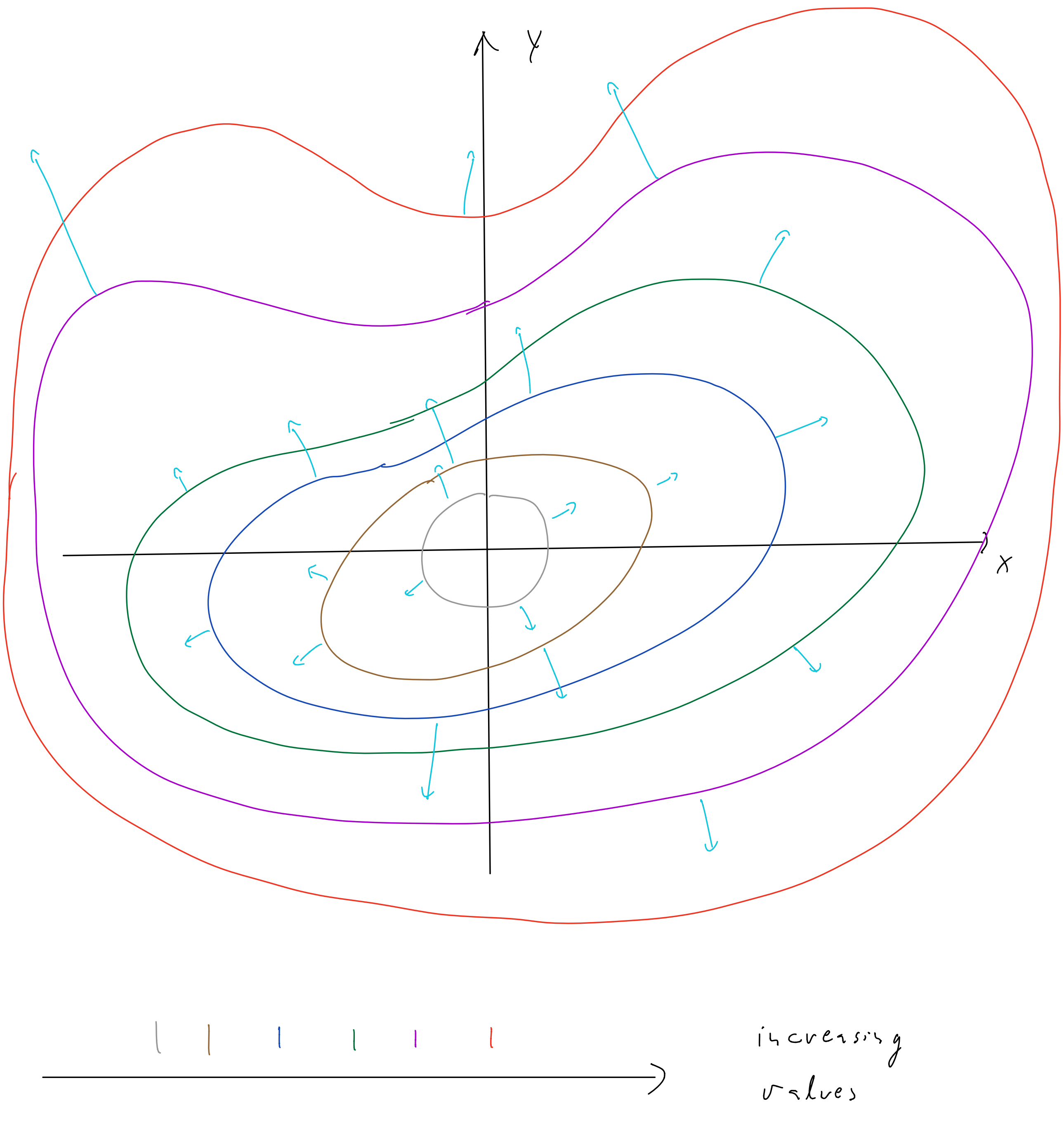
$$\left(x - \frac{1}{2}\right) + \left(y - \frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(z - \frac{1}{\sqrt{2}}\right) = 0.$$

The normal line is:

$$x - \frac{1}{2} = y - \frac{1}{2} = \frac{z - 1/\sqrt{2}}{1/\sqrt{2}}.$$

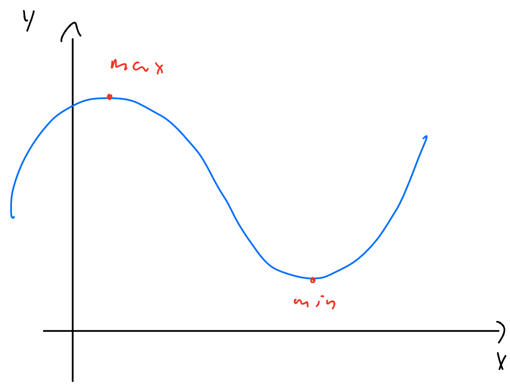
As a consequence of the above, the gradient ∇f is always orthogonal to the level surfaces of f. For 2 dimensions, the gradient is always orthogonal to the level curves of f. Moreover, ∇f points in the direction of maximal increase of the level curves/surfaces. Thus, from a plot of the level curves/surfaces, we can draw the gradient vector at every point, and such a plot is called a gradient vector field plot.

Ex: the level curves of $f = f(x,y)$ are given below. Sketch the gradient vector field.

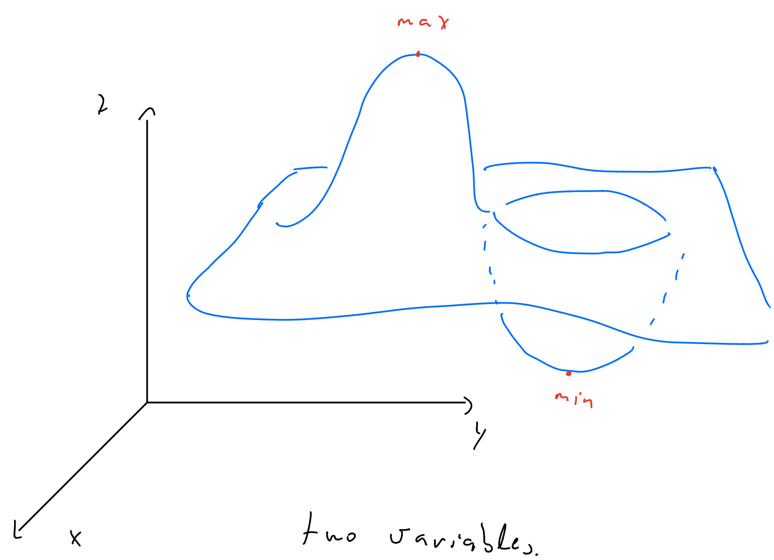


Maximum and minimum values

We will now study maxima and minima of functions $f = f(x,y)$, generalizing what we know from single variable calculus



one variable.



two variables.

Def. A function of two variables $f = f(x,y)$ has a local minimum at (x_0,y_0) if $f(x_0,y_0) \le f(x,y)$ for all (x,y) in a neighborhood of (x_0,y_0). The number $f(x_0,y_0)$ is then called a local minimum value. f has a local maximum at (x_0,y_0) if $f(x_0,y_0) \ge f(x,y)$ for all

(x,y) in a neighborhood of (x_0,y_0). The number $f(x_0,y_0)$ is then called a local maximum value. If $f(x_0,y_0) \leq f(x,y)$ for all (x,y) in the domain of f then we say that f has an absolute minimum at (x_0,y_0), and $f(x_0,y_0)$ is called an absolute minimum value. If $f(x_0,y_0) \geq f(x,y)$ for all (x,y) in the domain of f then we say that f has an absolute maximum at (x_0,y_0), and $f(x_0,y_0)$ is called and absolute maximum value. A point (x_0,y_0) is called a critical point of f if $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, or if one of thes partial derivatives does not exist.

Note that every absolute max/min is also a local max/min, but the converse is not true.

Like in single variable calculus, we can use derivatives to investigate maxima and minima.

First derivative test. If f has a local max or min at (x_0,y_0) and $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist, then $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$.

Note that a max/min can exist without the derivatives existing. E.g., $f(x,y) = \sqrt{x^2+y^2}$ has an absolute minimum at $(0,0)$, but the partial derivatives do not exist at $(0,0)$.

Second derivative test. Suppose that the second partial derivatives of f are continuous on a disk centered at (x_0,y_0) and suppose that $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$. Let

$$D = D(x_0,y_0) = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$$

$$= \det \begin{bmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix}.$$

(i) If $D > 0$ and $f_{xx}(x_0,y_0) > 0$ then $f(x_0,y_0)$ is a local min.
(ii) If $D > 0$ and $f_{xx}(x_0,y_0) < 0$ then $f(x_0,y_0)$ is a local max.
(iii) If $D < 0$, then $f(x_0,y_0)$ is not a local max or min.

<u>Remark.</u> In case (iii), we say that (x_0, y_0) is a saddle point. The graph of f then crosses its tangent plane at (x_0, y_0). If $D = 0$, then the test gives no information: f could have a local max, min, or neither at (x_0, y_0).

<u>Ex</u>: Find the local max, min, and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Compute

$$f_x(x,y) = 4x^3 - 4y \quad , \quad f_y(x,y) = 4y^3 - 4x.$$

Setting equal to zero:

$$\begin{cases} x^3 - y = 0, \\ y^3 - x = 0. \end{cases}$$

Plugging $y = x^3$ into the second equation:

$$x^9 - x = 0,$$

$$x(x^8 + 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

$$\Rightarrow \quad x = -1, 0, 1.$$

Since $y = x^3$ we find

$x = -1 \Rightarrow y = (-1)^3 = -1$, so $(-1,-1)$ is a critical point.

$x = 0 \Rightarrow y = 0^3 = 0$, so $(0,0)$ is a critical point.

$x = 1 \Rightarrow y = 1^3 = 1$, so $(1,1)$ is a critical point.

Next

$f_{xx}(x,y) = 12x^2$, $f_{xy}(x,y) = -4$, $f_{yy}(x,y) = 12y^2$.

$D(x,y) = 12x^2 \cdot 12y^2 - 4^2 = 144x^2y^2 - 16$

Then

$D(-1,-1) = 128 > 0$, $f_{xx}(-1,-1) = 12 > 0$, so f has a local min at $(-1,-1)$. The local min value is $f(-1,-1) = -1$.

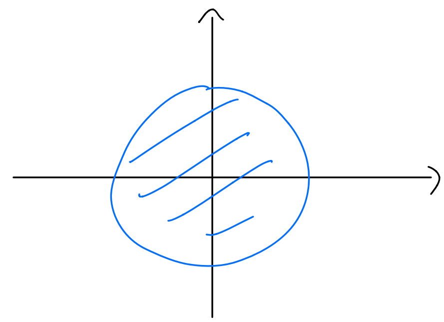
$D(0,0) = -16 < 0$, so f has a saddle point at $(0,0)$.

$D(1,1) = 128 > 0$, $f_{xx}(1,1) = 12 > 0$, so f has a local min at $(1,1)$. The local min value is $f(1,1) = -1$.

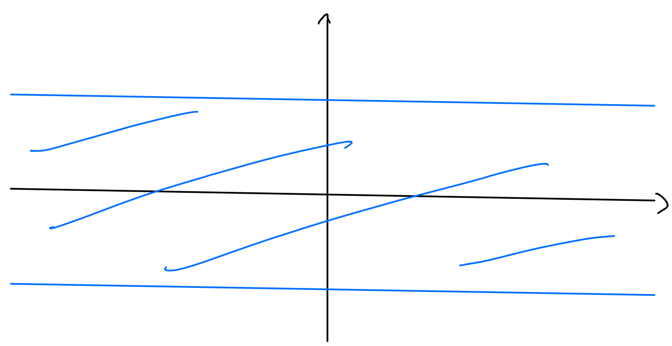
Absolute maximum and minimum values

A closed set in $\mathbb{R}^2$ is one that contains all its boundary points. For example, the sets

(a) $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

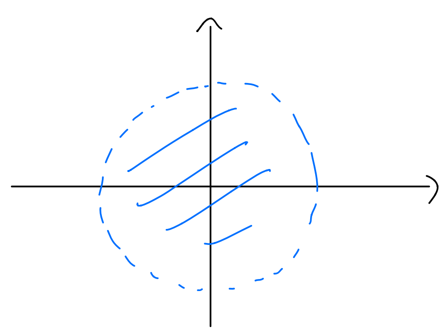


(b) $\{(x,y) \in \mathbb{R}^2 \mid |y| \leq 1\}$

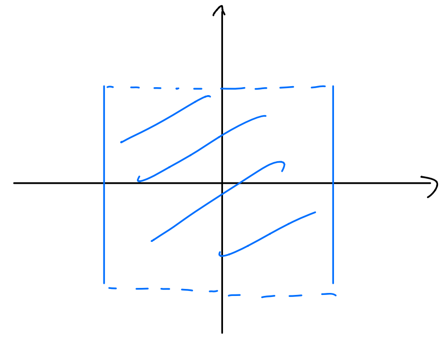


are closed, while the sets

(c) $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$



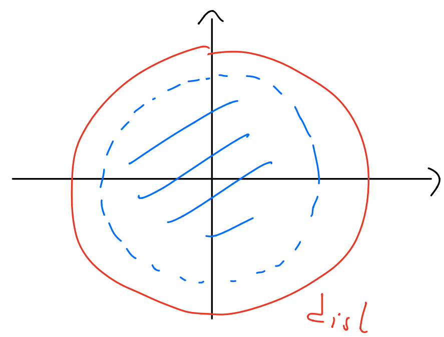
(d) $\{(x,y) \in \mathbb{R}^2 \mid |x| \le 1, |y| < 1\}$



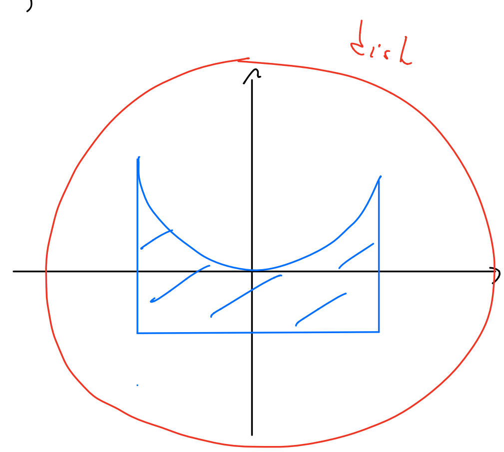
are not closed.

A **bounded set** in $\mathbb{R}^2$ is one that is contained in some disk. For example, the sets

(e) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$



(f) $\{(x,y)\ \mathbb{R}^2 \mid -1 \le x \le 1, -1 \le y \le x^2\}$



are bounded.

The set (b) above is not bounded.

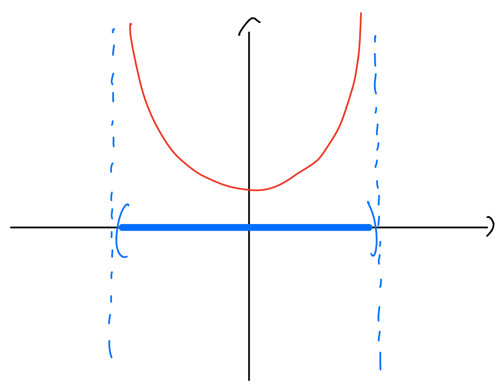
For the next result, we will consider sets that are closed and bounded. In the above examples:

	closed	bounded	closed and bounded
(a)	✓	✓	✓
(b)	✓	✗	✗
(c)	✗	✓	✗
(d)	✗	✓	✗
(e)	✗	✓	✗
(f)	✓	✓	✓

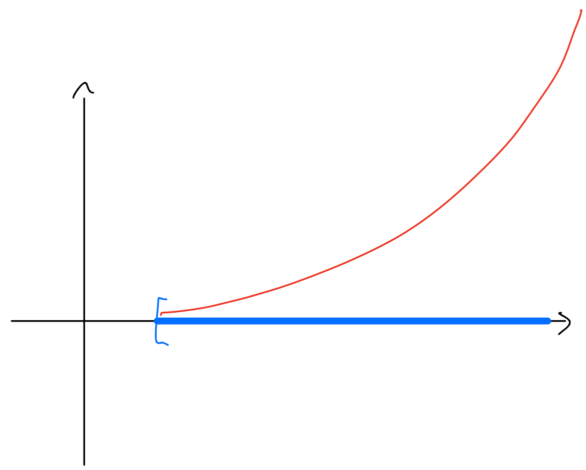
Extreme value theorem for functions of two variables.

If f is closed on a closed and bounded set $D \subset \mathbb{R}^2$ then f attains an absolute maximum value $f(x_1, y_1)$ and and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2), respectively.

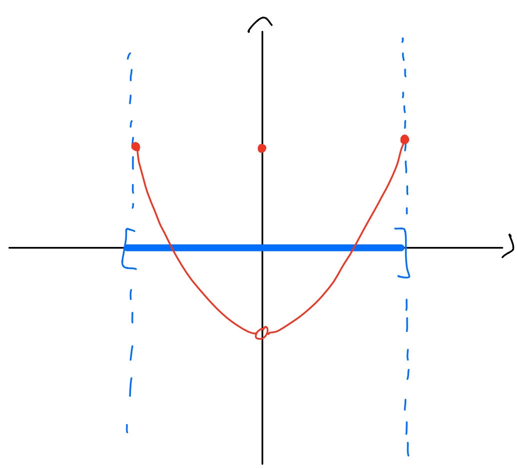
This is the two-dimensional analogue of a similar result in single-variable calculus.



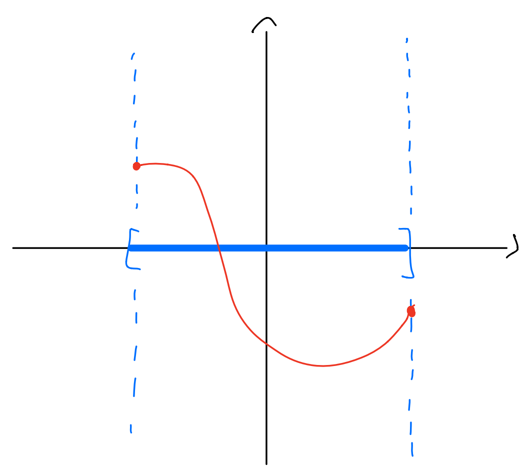
Set bounded but not closed, f has no absolute max.



set closed but not bounded, f has no absolute max.

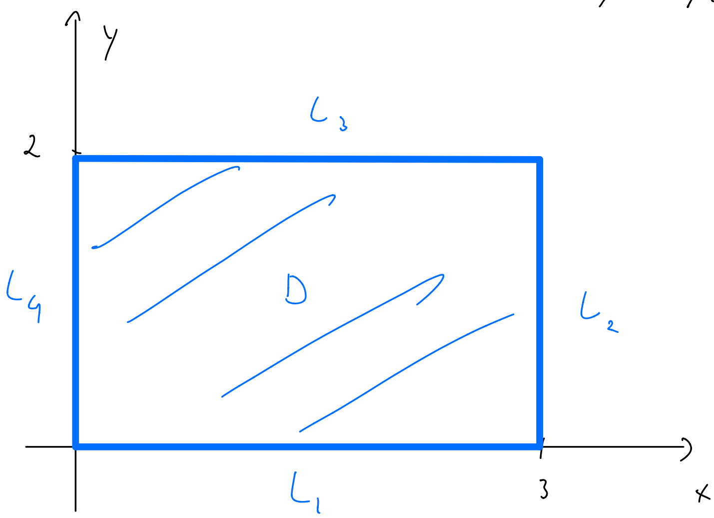


Closed and bounded set but f not continuous, f does not attain an absolute min.



closed and bounded set and f continuous, f attains an absolute max and a min.

Ex: Find the absolute max and min of the function $f(x,y) = x^2 - 2xy + 2y$ on $D = \{(x,y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.



First, note that D is bounded and closed and f is continuous, so we know from the extreme value theorem that f attains a maximum and a minimum.

Let us first find the critical points of f.

$$\left.\begin{aligned} f_x(x,y) &= 2x - 2y = 0 \\ f_y(x,y) &= -2y + 2 = 0 \end{aligned}\right\} \Rightarrow y = 1, \; x = 1.$$

The value of f at $(1,1)$ is $f(1,1) = 1$.

Since the max/min can be on the boundary, we consider f along the boundary of D, which is given by the four lines

$$L_1 = \{(x,0) \mid 0 \le x \le 3\}$$

$$L_2 = \{(3,y) \mid 0 \le y \le 2\}$$

$$L_3 = \{(x,2) \mid 0 \le x \le 3\}$$

$$L_4 = \{(0,y) \mid 0 \le y \le 2\}$$

On L_1, $f(x,y) = f(x,0) = x^2$, $0 \le x \le 3$. This is now a function of one variable that has a minimum at $x = 0$ and a max at $x = 3$, giving

$$f(0,0) = 0, \quad f(3,0) = 9.$$

On L_2, $f(x,y) = f(3,y) = 9 - 4y$, $0 \le y \le 2$. This is now a function of one variable that has a max at $y = 0$ and a min at $y = 2$, giving

$$f(3,0) = 9, \quad f(3,2) = 1.$$

On L_3, $f(x,y) = f(x,2) = x^2 - 4x + 4 = (x-2)^2$, $0 \le x \le 3$. This is now a function of one variable that has a max at $x = 0$ and a min at $x = 2$, giving

$$f(0,2) = 4, \quad f(2,2) = 0.$$

On L_4, $f(x,y) = f(0,y) = 2y$, $0 \le y \le 2$. This is now a function of one variable that has a max at $y = 2$ and a min at $y = 0$, giving

$$f(0,2) = 4, \quad f(0,0) = 0.$$

Now we consider all the values of f at the critical points and max and min on the boundary:

$f(1,1) = 1$, $f(0,0) = 0$, $f(3,0) = 9$, $f(3,0) = 9$, $f(3,2) = 1$, $f(0,2) = 4$, $f(2,2) = 0$, $f(0,2) = 4$, $f(0,0) = 0$.

The largest value among these is the absolute max and the smallest the absolute min. Thus, the absolute max is 9, attained at (3,0), and the absolute min is 0, attained at (0,0) and (2,2).

Note that the absolute max/min can be attained at more than one point.

Finding the absolute maximum and minimum value of a continuous function f on a closed and bounded set D.

1. Find the values of f at the critical points.
2. Find the extreme values of f on the boundary of D.
3. The largest/smallest of the values in steps 1 and 2 are the absolute max/min.

Lagrange multipliers

The volume of a box with dimensions x, y, z is $V(x,y,z) = xyz$. The volume has no maximum and the minimum is 0, i.e., a "box" that is a point (said differently, if we consider only $x, y, z > 0$ then V has no minimum). Suppose, however, that we want to find the largest possible volume such that the area of the box is fixed equal to a given value, say 12. Then we want to maximize

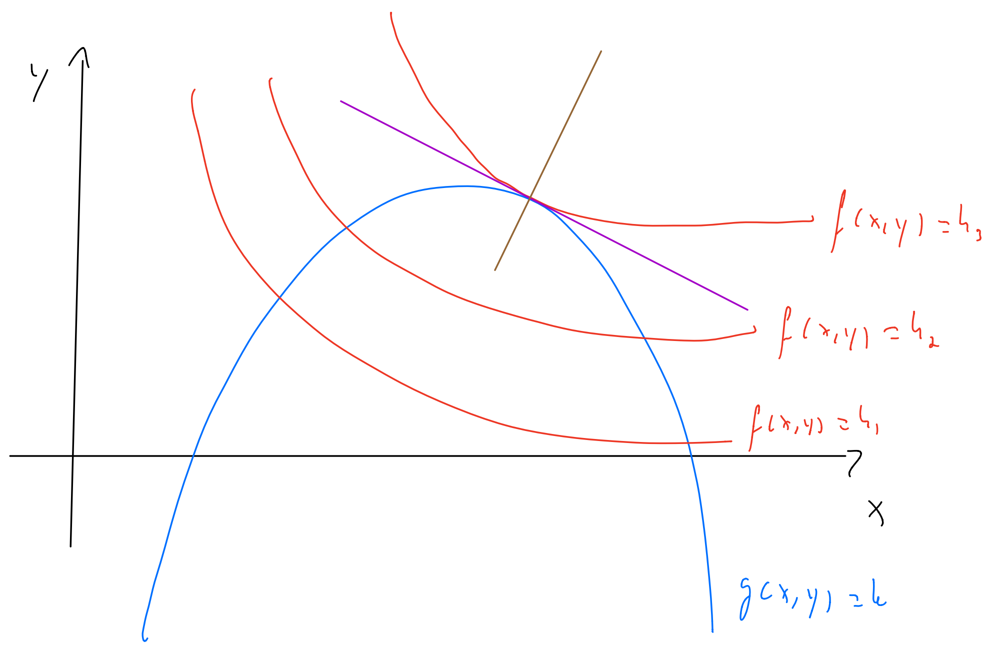
$$V(x,y,z) = xyz$$

subject to the condition

$$2xy + 2xz + 2yz = 12.$$

A condition of this form, that imposes a relation between the independent variables is called a <u>constraint</u>. The method of <u>Lagrange multipliers</u> is used to maximize or minimize a function subject to a constraint.

In order to understand the method, consider first the case of two variables and suppose we want to maximize $f(x,y)$ subject to the constraint $g(x,y)=k$. This means that if we consider the level curves $f(x,y)=h$ for different values of h, we are at intersections of $f(x,y)=h$ and $g(x,y)=k$, and among these intersections we want the one(s) for which h is the largest/smallest.



At an intersection point of $f(x,y)=h$ and $g(x,y)=k$ where f has a max/min, the curves $f(x,y)=h$

and $g(x,y)=k$ have to be tangent to each other, i.e., they have a common tangent line. Otherwise, we could change h a little bit to make $f(x,y)=h$ increase/decrease while still intersecting $g(x,y)=k$. Since ∇f and ∇g are orthogonal to the level curves of f and g, we conclude that they must be parallel, i.e., there exists a λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

at the point (x_0, y_0) where the level curves of f and g are tangent (provided $\nabla f(x_0, y_0) \neq (0,0)$).

The same idea can be applied to functions of more variables.

Method of Lagrange multipliers. To find the maximum and minimum values of $f = f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ (assuming these values exist and $\nabla g(x,y,z) \neq (0,0,0)$ on the surface $g(x,y,z) = k$), proceed as follows:

(a) Find the points (x,y,z) such that

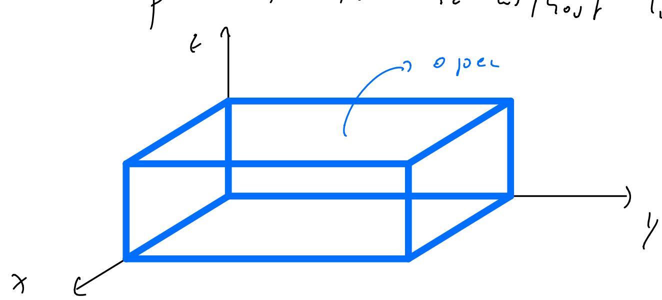
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and

$$g(x,y,z) = k.$$

(b) Evaluate f at the points found in (a). The largest value is the maximum of f and the smallest value is the minimum of f.

EX: What is the maximum volume a box with surface area equal to 12 and without lid?



We want to maximize

$$f(x,y,z) = xyz$$

subject to the constraint

$$g(x,y,z) = xy + 2xz + 2yz = 12.$$

We have

$$\nabla f = (yz,\ xz,\ xy)$$

$$\nabla g = (y+2z,\ x+2z,\ 2x+2y).$$

Thus $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ gives

$$\begin{cases} yz = \lambda(y+2z) \\ xz = \lambda(x+2z) \\ xy = \lambda(2x+2y) \end{cases}$$

We cannot have $\lambda = 0$, because this would give $yz = xz = xy = 0$, so $xy + 2xz + 2yz \neq 12$. To solve the system, multiply the first, second, and third equations by x, y, and z, respectively:

$$\begin{cases} yz = \lambda(y+2z) & \cdot x \\ xz = \lambda(x+2z) & \cdot y \\ xy = \lambda(2x+2y) & \cdot z \end{cases} \Rightarrow \begin{cases} xyz = \lambda(xy+2xz) \\ xyz = \lambda(xy+2yz) \\ xyz = \lambda(2xz+2yz) \end{cases}$$

Subtracting the first two equations

$$2xz - 2yz = 0 \Rightarrow x = y \quad (\text{since } z = 0 \text{ would give } f(x,y,z) = 0).$$

Subtracting the last two:

$$xy - 2xz = 0 \Rightarrow y = 2z \quad (x \neq 0)$$

Plugging $x = y = 2z$ into $xy + 2xz + 2yz = 12$ gives

$$4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = 1$$

($z = -1$ is not allowed because $x, y, z \geq 0$). Then $x = y = 2$.

The maximum volume is

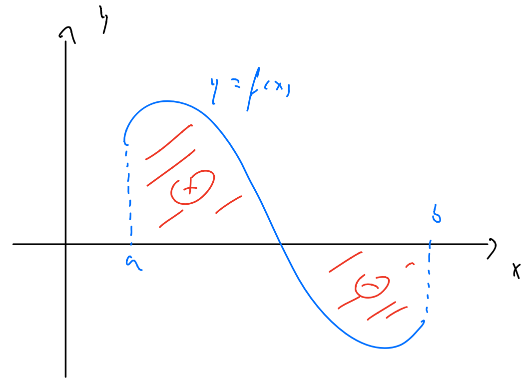
$$f(2,2,1) = 2 \cdot 2 \cdot 1 = 4.$$

Double integrals over rectangles

In single-variable calculus, the integral

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)\,\Delta x$$

represents the signed area under the graph of f between $x=a$ and $x=b$.



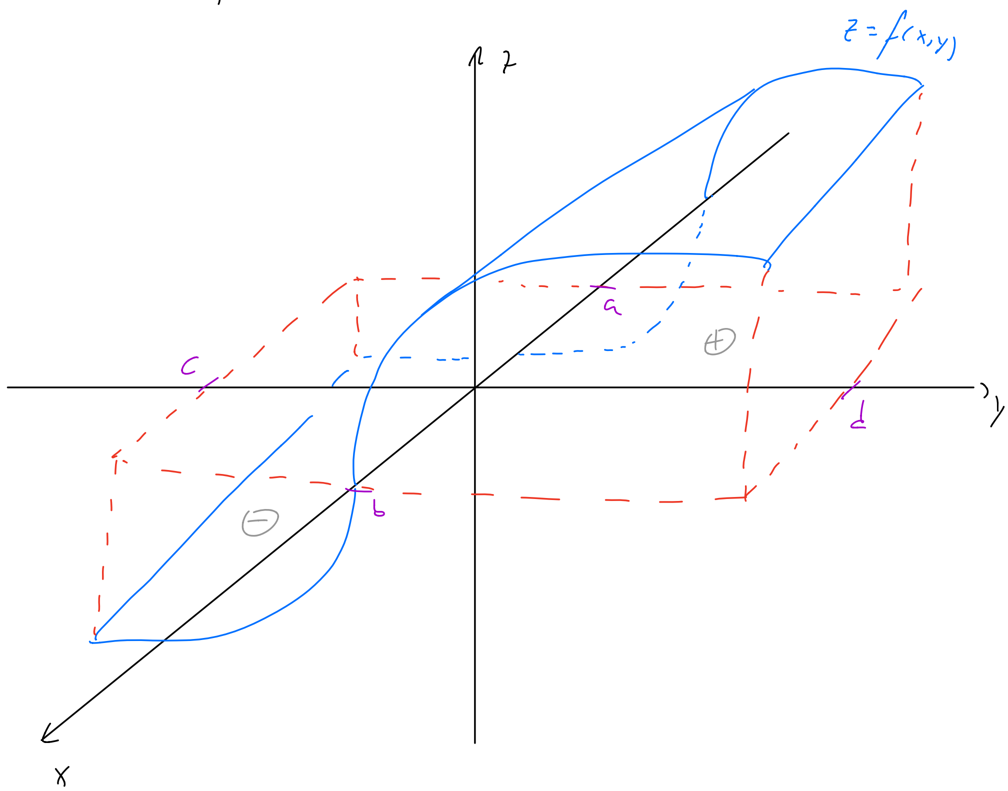
The limit is defined as the limit of Riemann sums $\sum_{i=1}^{N} f(x_i^*)\,\Delta x$, where $\Delta x = \frac{b-a}{N}$ and $x_i^* \in [x_{i-1}, x_i]$,

$x_0 = a$, $x_N = b$, $x_i = x_0 + i\,\Delta x$.

Similarly, if R is a rectangle in $\mathbb{R}^2$ given by

$$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\} = [a,b]\times[c,d],$$

and f is a function of two variables defined on R, We would like to define to define the integral of f over R such that it measures the signed volume under the graph of f in the region R.



To do so, we consider the rectangle $[a,b] \times [c,d]$ and set

$$\Delta x = \frac{b-a}{M}, \quad \Delta y = \frac{d-c}{N},$$

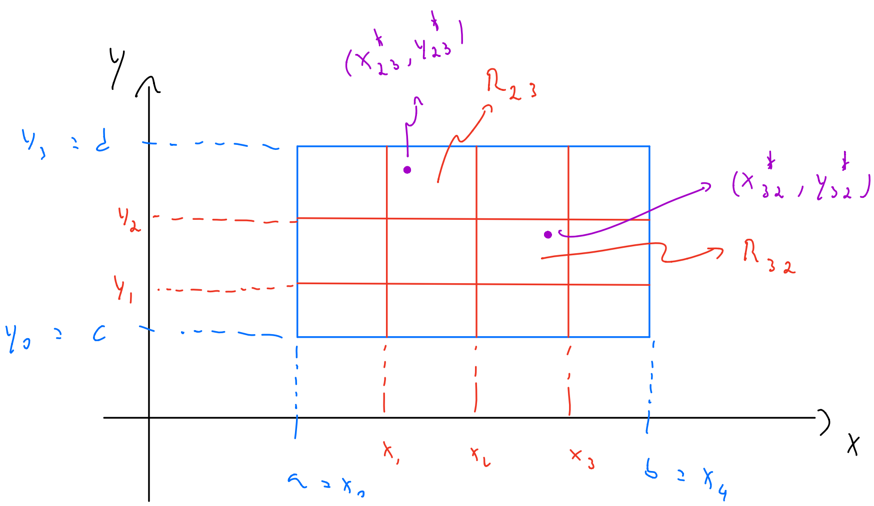
for some integers $M, N > 0$. Set $x_0 = a$, $y_0 = c$,

$$x_i = x_0 + i\Delta x, \quad y_j = y_0 + j\Delta y, \quad i = 1, \dots, M, \; j = 1, \dots, N.$$

For each i, j, consider the subrectangle

$$R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i], \quad i = 1, \dots, M, \ j = 1, \dots, N,$$

and a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$. Each R_{ij} has area $\Delta A = \Delta x \Delta y$. The subrectangles R_{ij} give a subdivision of R into MN subrectangles. The points (x_i^*, y_j^*) are called <u>sampling points.</u>



The sum

$$S_{MN} := \sum_{i=1}^{M} \sum_{j=1}^{N} f(x_{ij}^*, y_{ij}^*) \Delta A$$

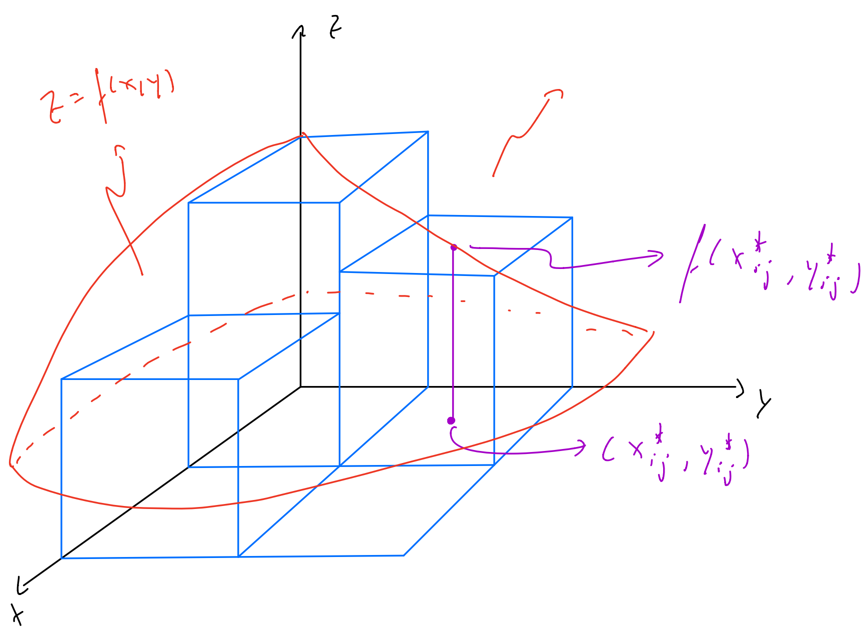
is called a <u>double Riemann sum.</u>

We define the double integral of f over the rectangle R as

$$\iint_R f(x,y)\,dA := \lim_{M,N\to\infty} S_{MN} = \lim_{M,N\to\infty} \sum_{i=1}^{M}\sum_{j=1}^{N} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

provided that the limit exists. If it does, we say that f is integrable in the region R.

The idea is very similar to single variable calculus: taking smaller rectangles R_{ij} we get better approximations for the volume under the graph, with an exact value in the limit.



As in single-variable calculus, we will learn ways of computing integrals that bypass evaluating the limit directly.

Iterated integrals

Suppose that $f = f(x,y)$ is defined on $R = [a,b] \times [c,d]$. For fixed $x \in [a,b]$, $f(x,y)$ is a function of y only, thus we can compute its integral w.r.t. to y:

$$\int_c^d f(x,y)\, dy .$$

This procedure is called partial integration w.r.t. y. We can do this for x fixed, but fixing different values of x yields different values for the integral $\int_c^d f(x,y)\, dy$. Thus, $\int_c^d f(x,y)\, dy$ is a function of x, i.e.,

$$A(x) = \int_c^d f(x,y)\, dy .$$

We can now integrate the function $A(x)$ w.r.t. x:

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx = \int_a^b \int_c^d f(x,y)\, dy\, dx ,$$

which is called a <u>iterated integral</u>. Alternatively, we can integrate first in x and then in y:

$$\int_c^d \left[\int_a^b f(x,y)\, dx \right] dy = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

The philosophy to compute iterated integrals is similar to partial derivatives: <u>if we integrate first in x, we treat y as constant; if we integrate first in y, we treat x as constant.</u>

<u>Ex</u>: Compute

$$\int_0^1 \int_0^2 x e^y \, dxy \quad , \quad \int_0^2 \int_0^1 x e^y \, dy\, dx.$$

$$\int_0^1 \int_0^2 x e^y \, dx\, dy = \int_0^1 e^y \int_0^2 x\, dx\, dy = \int_0^1 e^y \frac{x^2}{2}\Big|_0^2 dy = 2\int_0^1 e^y\, dy$$

$$= 2(e-1).$$

$$\int_0^2 \int_0^1 x e^y \, dy\, dx = \int_0^2 x \int_0^1 e^y\, dy\, dx = \int_0^2 x e^y \Big|_0^1 dx = (e-1)\int_0^2 x\, dx$$

$$= 2(e-1).$$

Both integrals agree. Will this always be the case? And what is the relation between iterated integrals and double integrals? The answer is given by:

Fubini's theorem. Assume that f is defined on the rectangle $R = [a,b] \times [c,d]$. Assume that f is continuous in R. Then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

The conclusion remains true if f is bounded and continuous except for a finite number of smooth curves where f is discontinuous, and the iterated integrals exist.

Ex: Find $\iint_R f(x,y)\,dA$ where $R = [-1,1] \times [0,1]$ and

$$f(x,y) = \begin{cases} 2, & x \geq 0. \\ 0, & x < 0. \end{cases}$$

$$\int_{-1}^{1} \int_0^1 f(x,y)\,dy\,dx = \int_{-1}^{0} \int_0^1 \underbrace{f(x,y)}_{=0}\,dy\,dx + \int_0^1 \int_0^1 \underbrace{f(x,y)}_{=2}\,dy\,dx = 2$$

$$\int_0^1 \int_{-1}^{1} f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-1}^{0} \underbrace{f(x,y)}_{=0}\,dx + \int_0^1 \underbrace{f(x,y)}_{=2}\,dx \right) dy = 2.$$

Since both iterated integrals exist, f is bounded and discontinuous only along $x=0$, Fubini's theorem gives $\iint_R f(x,y)\,dA = 2.$

Observe that

$$\iint_R g(x)\, h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy.$$

If $A(R)$ is the area of the rectangle R, the average of f over R is defined as

$$f_{ave} := \frac{1}{A(R)} \iint_R f(x,y)\, dA.$$

(Compare with single-variable calculus.)

Double integrals over general regions

We defined the integral of $f = f(x,y)$ over a rectangle R. Next, we want to define integrals over more general regions.

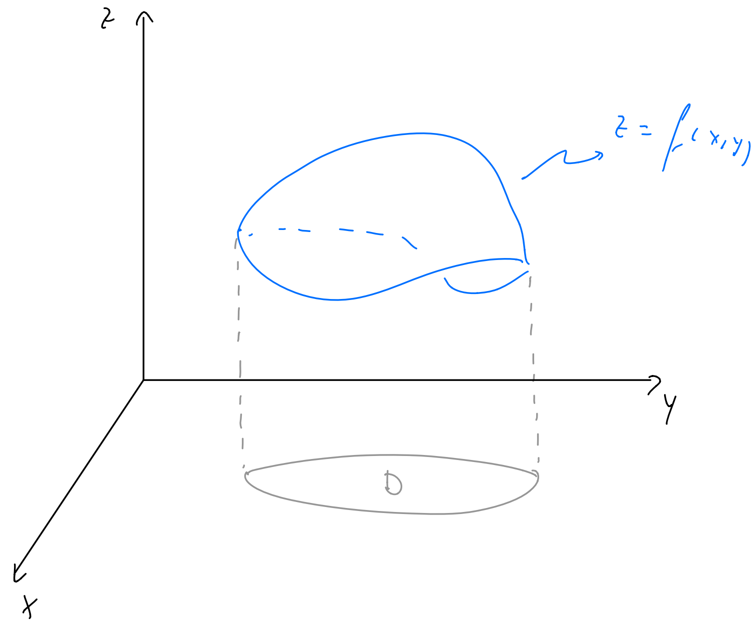
Let $f = f(x,y)$ be defined in a bounded region $D \subset \mathbb{R}^2$. Let R be a rectangle containing D and set

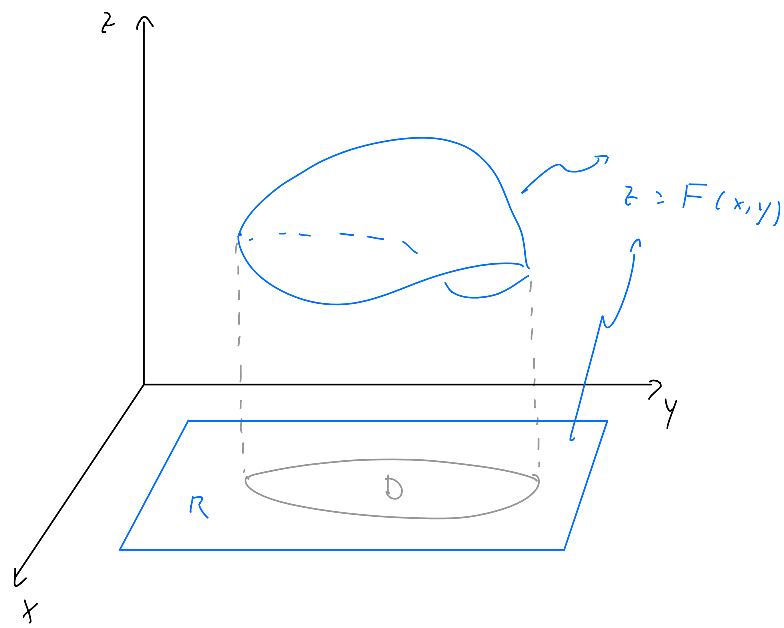
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \in R,\ (x,y) \notin D. \end{cases}$$

We define the double integral of f over D as

$$\iint_D f(x,y)\, dA := \iint_R F(x,y)\, dA.$$

The introduction of F is an artifact to recast the problem in terms of integrals over rectangles which have already been defined. But notice that $\iint_D f(x,y)\, dA$ in fact measures what is supposed to, namely, the signed volume under the graph of f in the region D.





We will next find a suitable generalization of iterated integrals to compute $\iint_D f(x,y)\,dA$ in practice.

Def. A region $D \subset \mathbb{R}^2$ is called a <u>region of type I</u> if it lies between the graphs of two continuous functions of x, i.e., if it is of the form

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

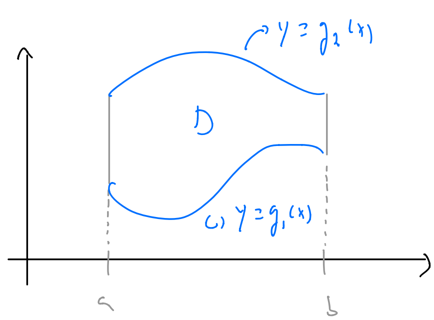
where g_1 and g_2 are continuous on $[a,b]$.

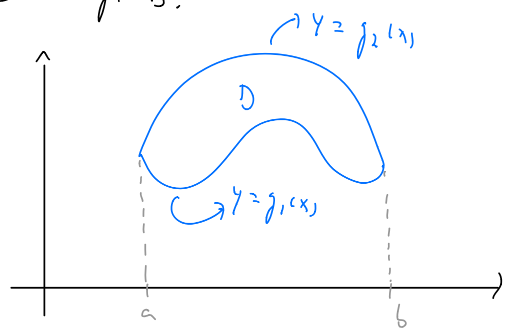
A region $D \subset \mathbb{R}^2$ is called a <u>region of type II</u> if it lies between the graphs of two continuous functions of y, i.e., if it is of the form

$$D = \{(x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\},$$

where h_1 and h_2 are continuous on $[c,d]$.

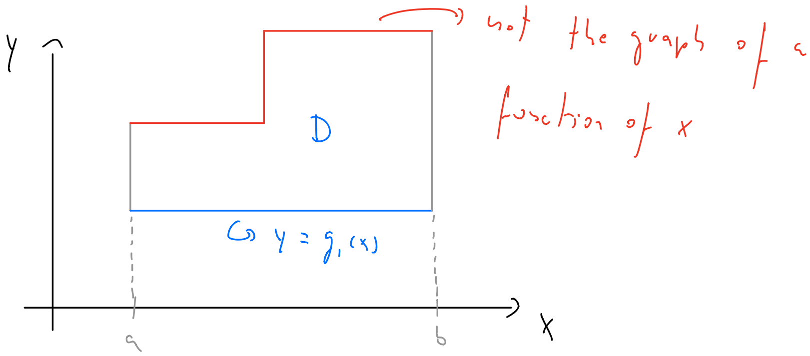
Ex: These are type I regions:



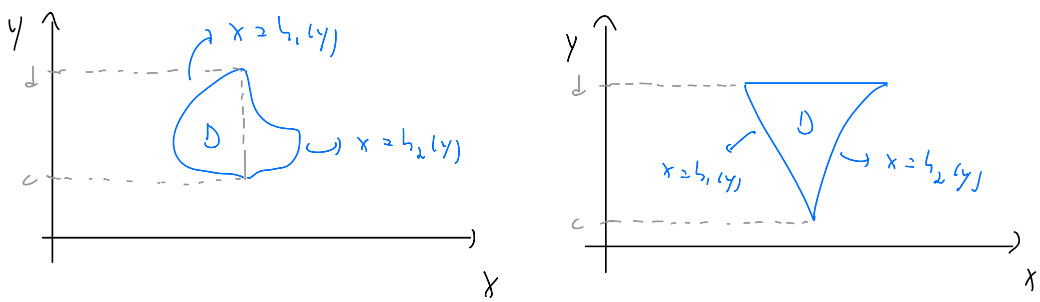


$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le x^3 + 1\}$$

The following is not a region of type I:



The following are type II regions:



$$D = \{(x,y) \in \mathbb{R}^2 \mid y \leq x \leq y+2, \ -1 \leq y \leq 1\}.$$

The non-type I region above is not of type II either.

If f is continuous on a region of type I

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

If f is continuous on a region of type II

$$D = \{ (x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y),\ c \le y \le d \}$$

then

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy .$$

Ex: Evaluate

$$\int_0^2 \int_0^{y^2} x^2 y \, dx\, dy \quad (\text{type II},\ h_1(y) = 0,\ h_2(y) = y^2)$$

$$\int_0^{\frac{\pi}{2}} \int_0^{x} x \sin y \, dy\, dx \quad (\text{type I},\ g_1(x) = 0,\ g_2(x) = x)$$

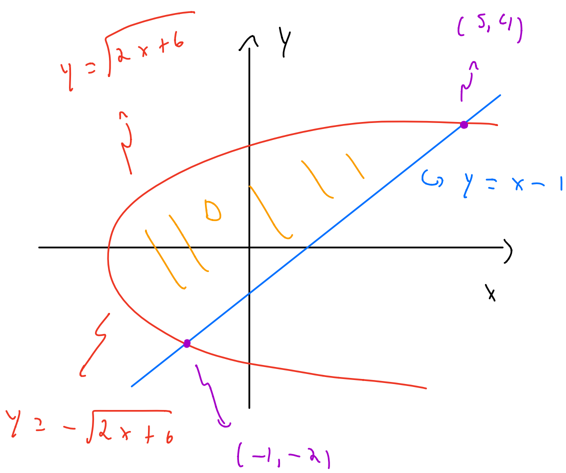
$$\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy = \int_0^2 y \underbrace{\int_0^{y^2} x^2 dx}_{= \frac{1}{3} x^3 \big|_0^{y^2} = \frac{1}{3} y^6} dy = \frac{1}{3} \int_0^2 y^7 dy = \frac{1}{24} y^8 \Big|_0^2 = \frac{256}{24} = \frac{32}{3}$$

$$\int_0^{\frac{\pi}{2}} x \underbrace{\int_0^x \sin y \, dy}_{= -\cos y \big|_0^x} dx = \int_0^{\frac{\pi}{2}} (-x\cos x + x)\, dx \underset{\text{by parts}}{=} \left[\frac{1}{2}x^2 - (x \sin x + \cos x) \right]_0^{\frac{\pi}{2}}$$

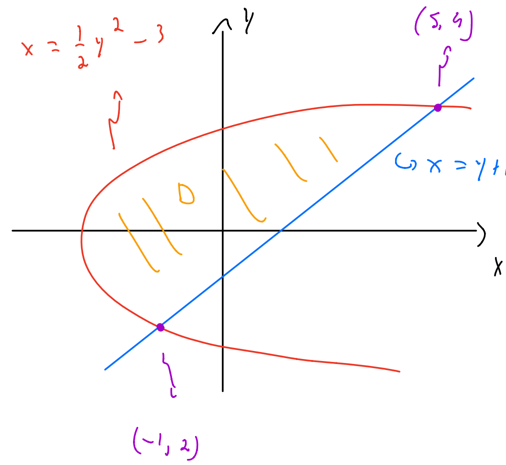
$$= \frac{\pi^2}{8} - \frac{\pi}{2} + 1 .$$

Note that some regions are both of type I and II, e.g.,

$$D = \left\{ (x,y) \mid \tfrac{1}{2}y^2 - 3 \le x \le y+1,\ -2 \le y \le 4 \right\}$$

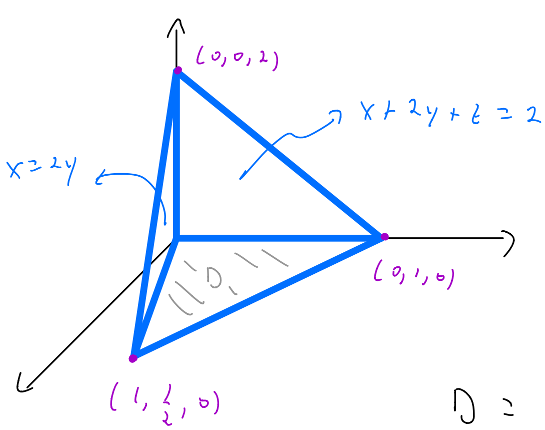


type I



type II

<u>Ex</u>: Write a double integral representing the volume of tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

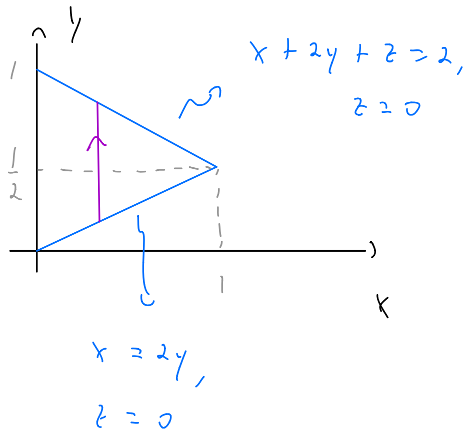


We want the volume under the graph of

$$z = f(x,y) = 2 - x - 2y$$

in the region

$$D = \left\{ (x,y) \mid 0 \le x \le 1,\ \frac{x}{2} \le y \le 1 - \frac{x}{2} \right\}$$



Then

$$V = \iint_D (2 - x - 2y)\, dA$$

$$= \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2 - x - 2y)\, dy\, dx.$$

Properties of double integrals.

(i) $\displaystyle\iint_D \big(f(x,y) + g(x,y)\big)\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

(ii) $\displaystyle\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA, \quad c \in \mathbb{R}$

(iii) If $f(x,y) \le g(x,y)$ for $(x,y) \in D$ then

$$\iint_D f(x,y)\, dA \le \iint_D g(x,y)\, dA$$

(iv) If $m \le f(x,y) \le M$ for $(x,y) \in D$ then

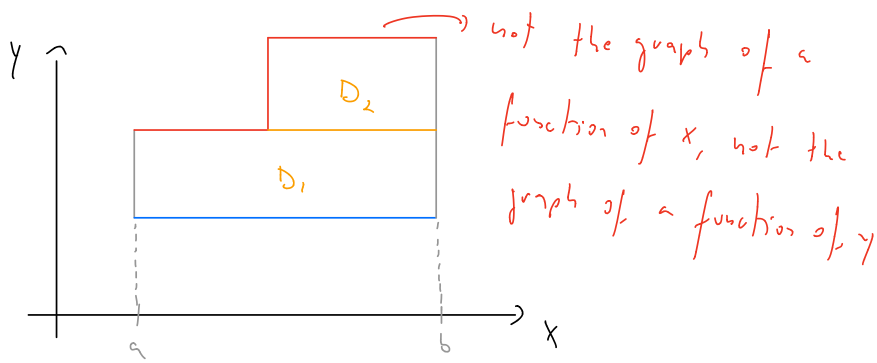
$$m \le \frac{1}{A(D)} \iint_D f(x,y)\, dA \le M,$$

where $A(D)$ = area of D.

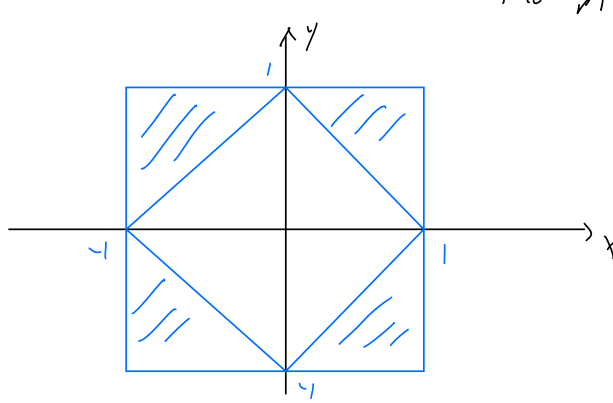
(v) If $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except possibly on their boundaries, then

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$

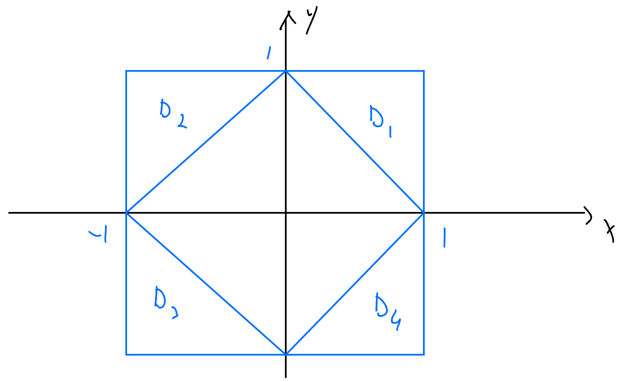
Property (v) is useful to compute integrals that are not of type I or II but are the union of type I or II regions.



Ex: Consider the region D in the picture



which is not of type I or II. But we can write



where

$$D_1 = \{ 0 \le x \le 1, \ -x+1 \le y \le 1 \}$$

$$D_2 = \{ -1 \le x \le 0, \ x+1 \le y \le 1 \}$$

$$D_3 = \{ -1 \le x \le 0, \ -1 \le y \le -x-1 \}$$

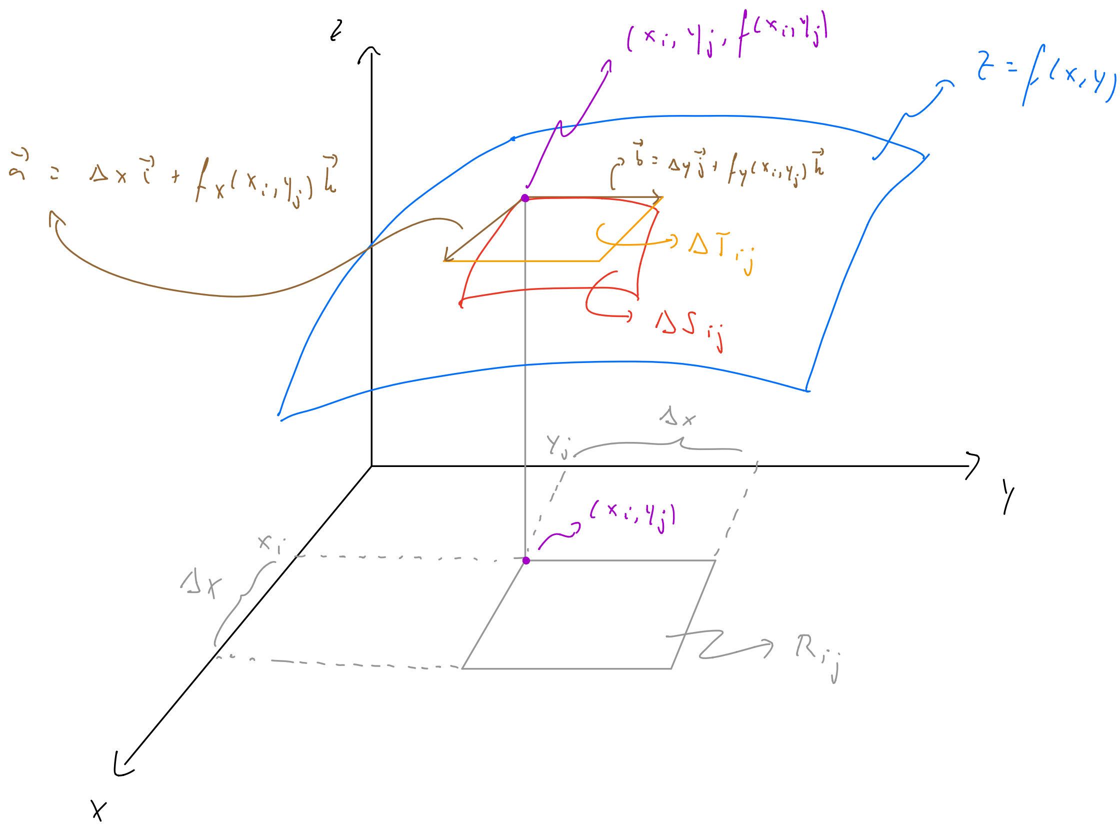
$$D_4 = \{ 0 \le x \le 1, \ -1 \le y \le x-1 \}$$

which are all of type I.

Surface area

Consider a surface in $\mathbb{R}^3$ given by a graph $z = f(x,y)$. How can we calculate its surface area?

Consider the construction illustrated in the picture (note that we are sampling with the points at the beginning of the intervals $[x_i, x_{i+1}]$, $[y_j, y_{j+1}]$).



We have

$$\text{area of } S = A(S) = \lim_{M,N\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} \Delta T_{ij}.$$

But

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & \Delta x f_x(x_i, y_j) \\ 0 & \Delta y & \Delta y f_y(x_i, y_j) \end{bmatrix}$$

$$= \left(-f_x(x_i, y_j)\,\vec{i} - f_y(x_i, y_j)\,\vec{j} + \vec{k}\right) \overbrace{\Delta x \Delta y}^{= \Delta A}$$

$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$

$$= \sqrt{(f_x(x_i, y_j))^2 + (f_y(x_i, y_j))^2 + 1}\;\Delta A$$

Hence

$$A(S) = \iint_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\; dA$$

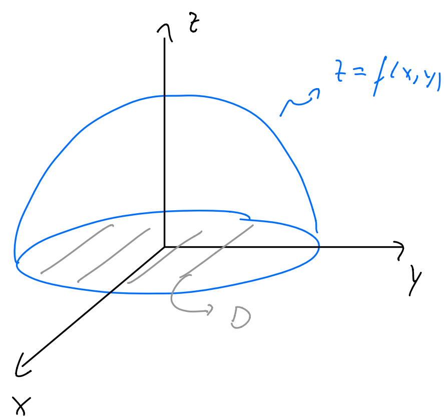
is a formula for the area of the surface. (Compare with the formula $L = \int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$ for the length of the graph in single-variable calculus.)

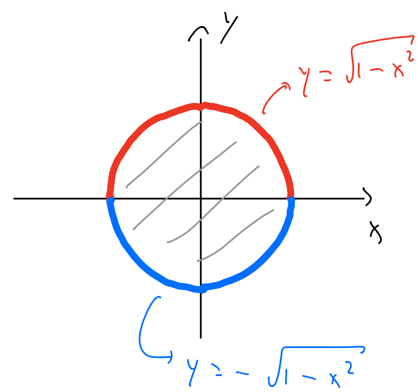
Ex: Write the area of a sphere of radius one as a double integral of type I.

The upper cap of the sphere is the graph of

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$

over the region $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$.





$$D = \left\{ (x,y) \;\middle|\; -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \right\}$$

$$\partial_x f(x,y) = -\frac{x}{\sqrt{1 - x^2 - y^2}}, \quad \partial_y f(x,y) = -\frac{y}{\sqrt{1 - x^2 - y^2}}$$

$$A = \underbrace{2}_{\text{two caps}} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{\frac{x^2 + y^2}{1 - x^2 - y^2} + 1}\; dy\, dx.$$

Triple integrals

Let $f = f(x,y,z)$ be defined in a box

$$B = \{ (x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s \}.$$

We can generalize the procedure employed to define double integrals to this case: set

$$\Delta x = \frac{b-a}{L}, \quad \Delta y = \frac{d-c}{M}, \quad \Delta z = \frac{s-r}{N},$$

$$x_i = x_0 + i\Delta x, \quad x_0 = a, \quad i = 1, \dots, L,$$

$$y_j = y_0 + j\Delta y, \quad y_0 = c, \quad j = 1, \dots, M,$$

$$z_k = z_0 + k\Delta z, \quad z_0 = r, \quad k = 1, \dots, N,$$

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

$$(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in B_{ijk},$$

$$\Delta V = \Delta x \Delta y \Delta z$$

and define the triple Riemann sum

$$S_{LMN} := \sum_{i=1}^{L} \sum_{j=1}^{M} \sum_{k=1}^{N} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

and the triple integral of f over B is

$$\iiint_B f(x,y,z)\,dV := \lim_{L,M,N\to\infty} S_{LMN}$$

$$= \lim_{L,M,N\to\infty} \sum_{i=1}^{L}\sum_{j=1}^{M}\sum_{k=1}^{N} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V.$$

These integrals can be computed with help of Fubini's theorem for triple integrals: if f is continuous on $B=[a,b]\times[c,d]\times[r,s]$ then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$= \text{other 5 possible orders of integration.}$$

Ex: Find $\iiint_B f(x,y,z)\,dV$ if $f(x,y,z) = x e^y z^2$ and $B = [0,2]\times[-1,1]\times[1,2]$.

$$\iiint_B f(x,y,z)\,dV = \int_1^2\int_{-1}^1\int_0^2 x e^y z^2\,dx\,dy\,dz = \int_1^2\int_{-1}^1 \frac{x^2}{2}\Big|_0^2 e^y z^2\,dy\,dz$$

$$= 2\int_1^2 e^y\Big|_{-1}^1 z^2\,dz = 2(e-e^{-1})\,\frac{z^3}{3}\Big|_1^2 = \frac{14}{3}(e-e^{-1}).$$

Let E be a bounded region in $\mathbb{R}^3$ and f a function defined in E. Let B be a box containing E and set

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

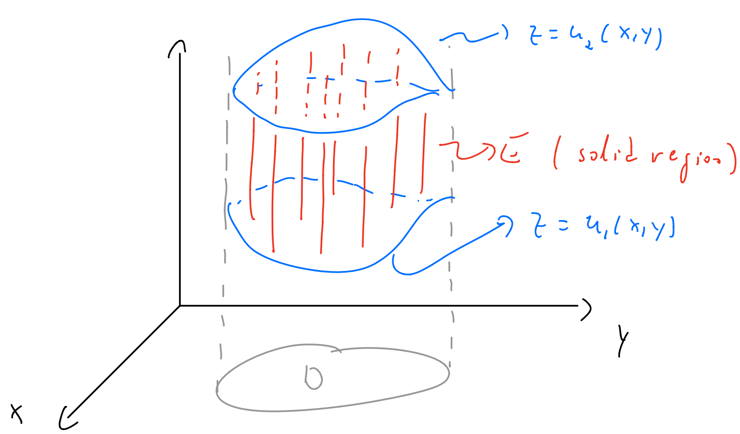
We define the <u>triple integral of f over E</u> as

$$\iiint_E f(x,y,z)\,dV := \iiint_B F(x,y,z)\,dV.$$

A region $E \subset \mathbb{R}^3$ is called of type 1 if it is of the form

$$E = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

for D a region in the xy-plane and u_1, u_2 continuous functions. Thus, E lies between the graphs of u_1 and u_2



If E is of type 1 and f is continuous, then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

$$= \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA.$$

Thus, we perform the integral in z between $u_1(x,y)$ and $u_2(x,y)$, treating x and y as constants, and then the integral over D.

In particular, if D is of type I, then

$$\iiint_V f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx,$$

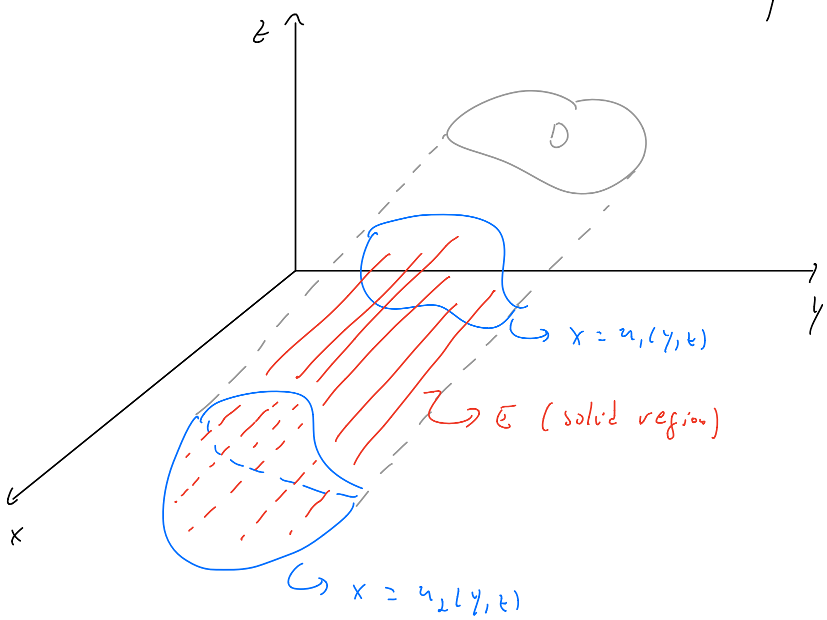
and if D is of type II, then

$$\iiint_V f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

A region $E \subset \mathbb{R}^3$ is called of type 2 if it is of the form

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; (y,z) \in D,\ u_1(y,z) \leq x \leq u_2(y,z) \right\}$$

for D a region in the yz-plane and continuous functions u_1 and u_2.



If f is continuous in E then

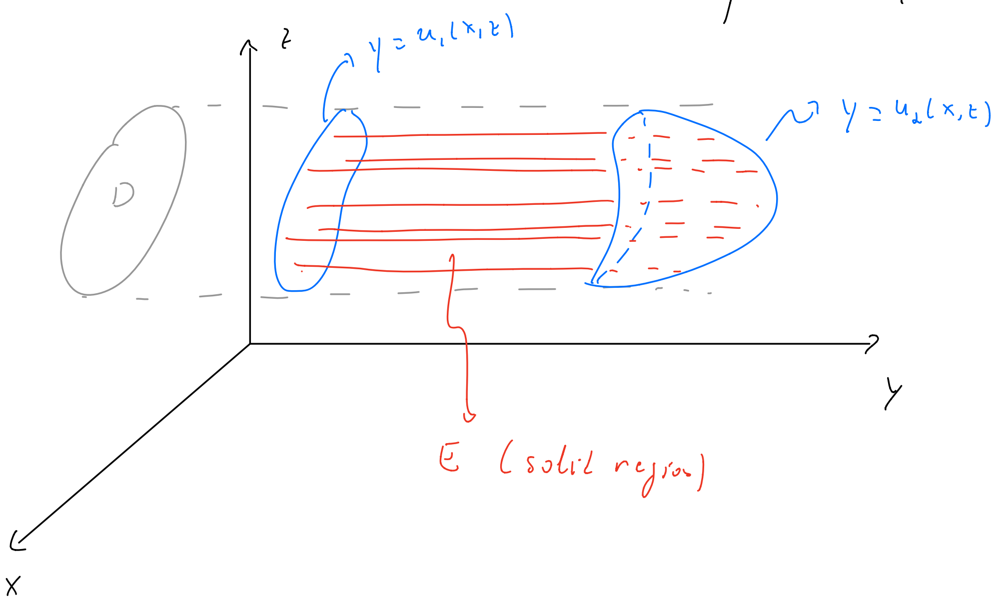
$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA$$

$$= \iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx\, dA.$$

A region $E \subset \mathbb{R}^3$ is called of type 3 if it is of the form

$$E = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,z) \in D,\ u_1(x,z) \leq y \leq u_2(x,z) \}$$

for D a region in the xz-plane and continuous functions u_1 and u_2.



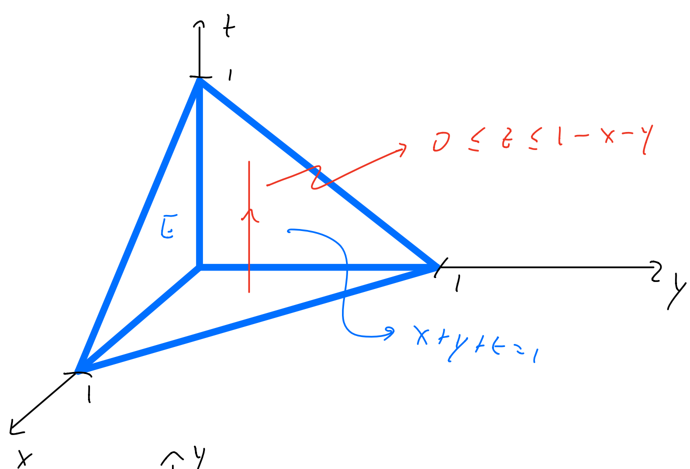
If f is continuous in E then

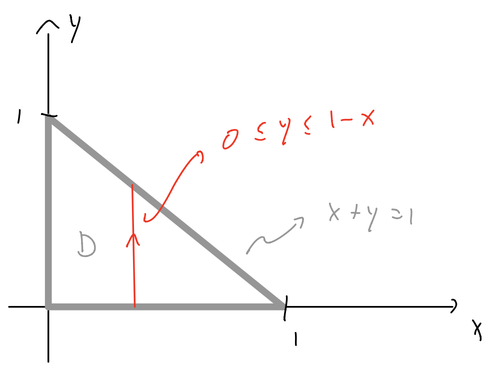
$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA$$

$$= \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy\, dA.$$

Ex: Find $\iiint_E f(x,y,z)\,dV$ where E is the region bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$, and $f(x,y,z)=z$.

For triple integrals, it is useful to draw two pictures, one for the region E and one for the region D.





Here we chose D on the xy-plane. Then

$$E = \{ (x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y \}$$

$$D = \{ (x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1-x \}.$$

$$\iiint_E f(x,y,z)\, dV = \iint_D \int_0^{1-x-y} f(x,y,z)\, dz\, dA$$

$$= \iint_D \int_0^{1-x-y} z\, dz\, dA = \iint_D \frac{z^2}{2} \Big|_0^{1-x-y} dA = \iint_D \frac{(1-x-y)^2}{2}\, dA$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2\, dy\, dx = \frac{1}{2} \int_0^1 -\frac{(1-x-y)^3}{3} \Big|_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left(-\frac{0}{3} + \frac{(1-x)^3}{3} \right) dx = \frac{1}{6} \int_0^1 (1-x)^3\, dx = \frac{1}{6} \left(-\frac{(1-x)^4}{4} \right) \Big|_0^1 = \frac{1}{24}.$$

Sometimes it is useful to write

$$\iint_D \int_0^{1-x-y} z\, dz\, dA = \iint_D \frac{z^2}{2} \Big|_{z=0}^{z=1-x-y} dA$$

$$\int_0^1 \int_0^{1-x} (1-x-y)^2\, dy\, dx = \int_0^1 -\frac{(1-x-y)^3}{3} \Big|_{z=0}^{z=1-x} dx,$$

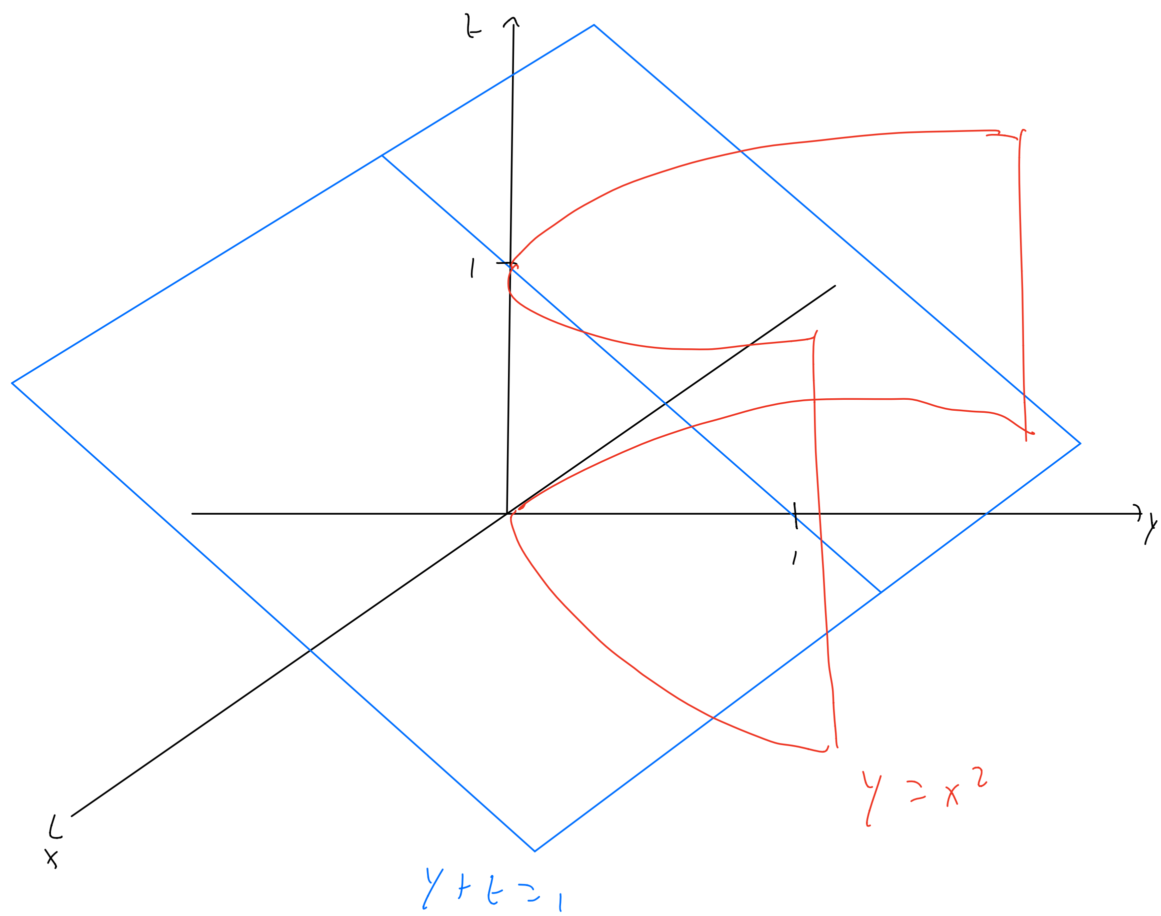
etc. to keep track of which variable is being substituted.

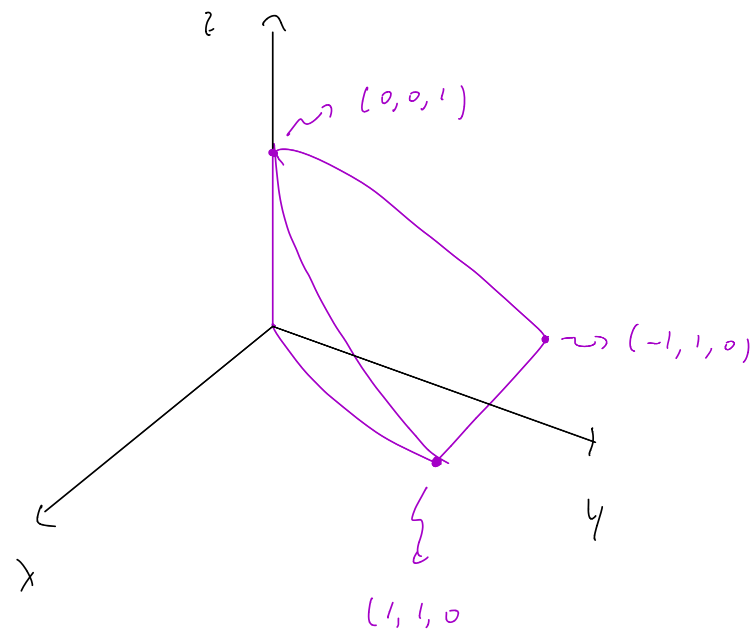
Volumes

The volume of a region $E \subset \mathbb{R}^3$ can be computed by

$$V(E) = \iiint_E dV.$$

Ex: Find the volume of the solid enclosed by $y = x^2$, $z = 0$, and $y + z = 1$





Thus

$$E = \{ -1 \le x \le 1, \quad x^2 \le y \le 1, \quad 0 \le z \le 1-y \}$$

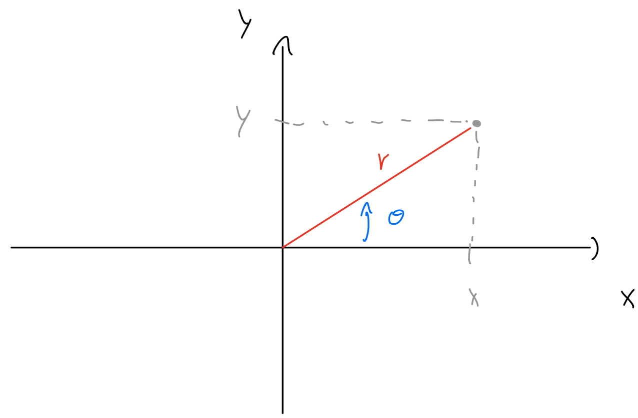
$$V = \iiint_E dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} (1-y)\,dy\,dx$$

$$= \int_{-1}^{1} \left(y - \frac{y^2}{2} \right)\Bigg|_{y=x^2}^{y=1} dx = \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx$$

$$= \left(\frac{1}{2}x - \frac{1}{3}x^3 + \frac{x^5}{10} \right)\Bigg|_{-1}^{1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{8}{15}.$$

Triple integrals in cylindrical coordinates

A point (x,y) in the xy-plane can be uniquely identified by its polar coordinates (r, θ)



where

$$r = \sqrt{x^2+y^2}$$

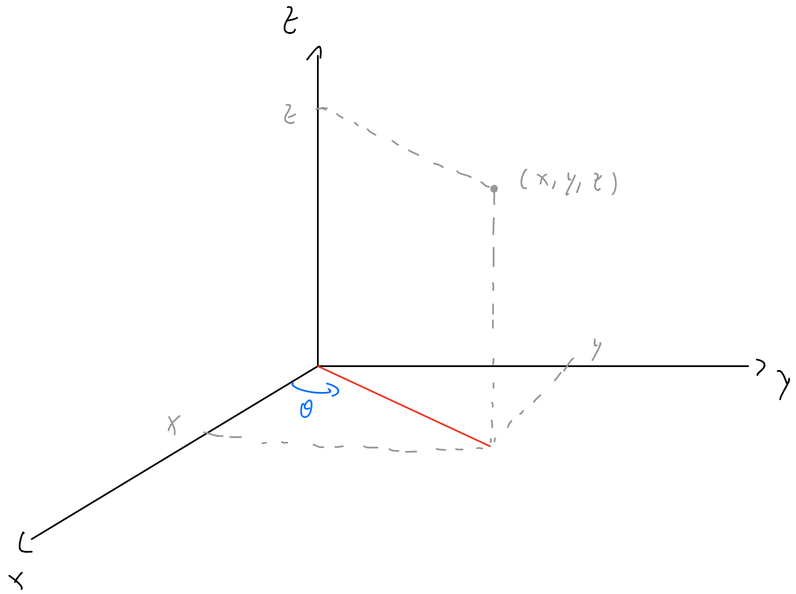
$$\tan\theta = \frac{y}{x}$$

A point (x,y,z) in $\mathbb{R}^3$ can be uniquely identified by its cylindrical coordinates (r, θ, z), where

$$r = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{y}{x}$$

$$z = z$$



Ex: Find cylindrical coordinates for the point with $\underbrace{\text{rectangular coordinates}}$ $(3, -3, 7)$.

i.e., the "standard" (x, y, z) coordinates, a.k.a. Cartesian coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{9 + 9} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1$$

There are infinitely many values of θ such that $\tan\theta = -1$, namely, $\theta = \frac{7\pi}{4} + 2n\pi$, $n \in \mathbb{Z}$. We can take $n = 0$, then $(3\sqrt{2}, \frac{7\pi}{4}, 7)$ are cylindrical coordinates for $(3, -3, 7)$ (as are $(3\sqrt{2}, \frac{7\pi}{4} \pm 2\pi, 7)$, $(3\sqrt{2}, \frac{7\pi}{4} \pm 4\pi, 7)$, etc.)

Ex: Find the Cartesian coordinates of the point given in cylindrical coordinates by $(2, \frac{\pi}{6}, 3)$.

The formula for (x, y, z) in terms of (r, θ, z) is

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Thus,
$$x = 2\cos\frac{\pi}{6} = 2\frac{\sqrt{3}}{2} = \sqrt{3},$$

$$y = 2\sin\frac{\pi}{6} = 2\cdot\frac{1}{2} = 1,$$

$$z = 3,$$

so $(\sqrt{3}, 1, 3)$ are the Cartesian coordinates.

If E is a region described by

$$E = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D, \ u_1(x,y) \le z \le u_2(x,y)\}$$

where D is given in polar coordinates by

$$D = \{(x,y) \in \mathbb{R}^2 \mid \alpha \le \theta \le \beta, \ h_1(\theta) \le r \le h_2(\theta)\}$$

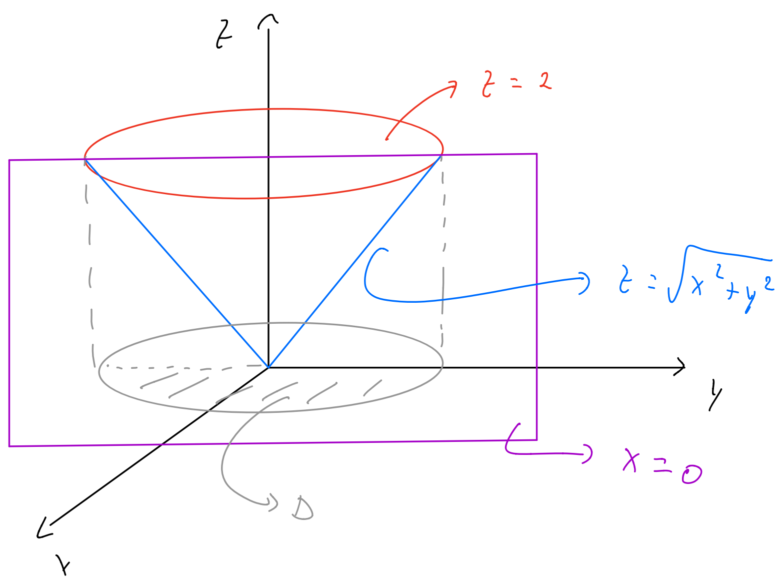
then

$$\iiint_V f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

which is known as the triple integral of f in cylindrical coordinates.

Ex: Find $I = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$.

This is the volume between the graphs of $z = \sqrt{x^2+y^2}$, $z = 2$, and the plane $x = 0$



Thus

$$I = \iiint_E f(x,y,z)\, dV$$

$f(x,y,z) = x^2+y^2$.

where

$$E = \{ \underbrace{0 \le x \le 2, \ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}}_{D}, \ x^2+y^2 \le z \le 2 \}$$

The region D is given in polar coordinates by

$$D = \{ (r, \theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2 \}$$

Since $x^2+y^2 = r^2$, E is described in cylindrical coordinates by

$$E = \{ (r, \theta, z) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2, \ r^2 \le z \le 2 \}$$

Thus $f(x,y,z) = f(x\cos\theta, y\sin\theta, z) = r^2$ and

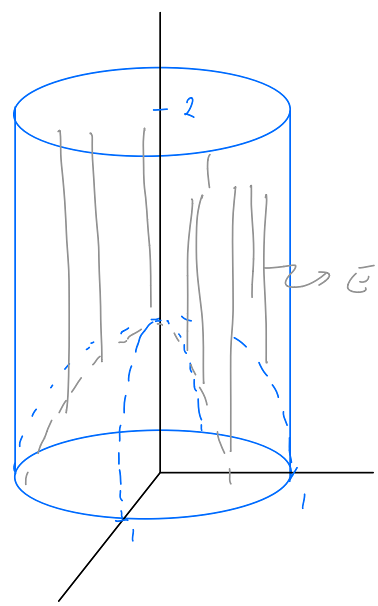
$$I = \iiint_E f(x,y,z)\,dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \int_r^2 r^2 \, r \, dz\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 r^3(2-r)\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}r^4 - \frac{r^5}{5}\right)\Big|_0^2 d\theta$$

$$= \frac{8}{5}\pi.$$

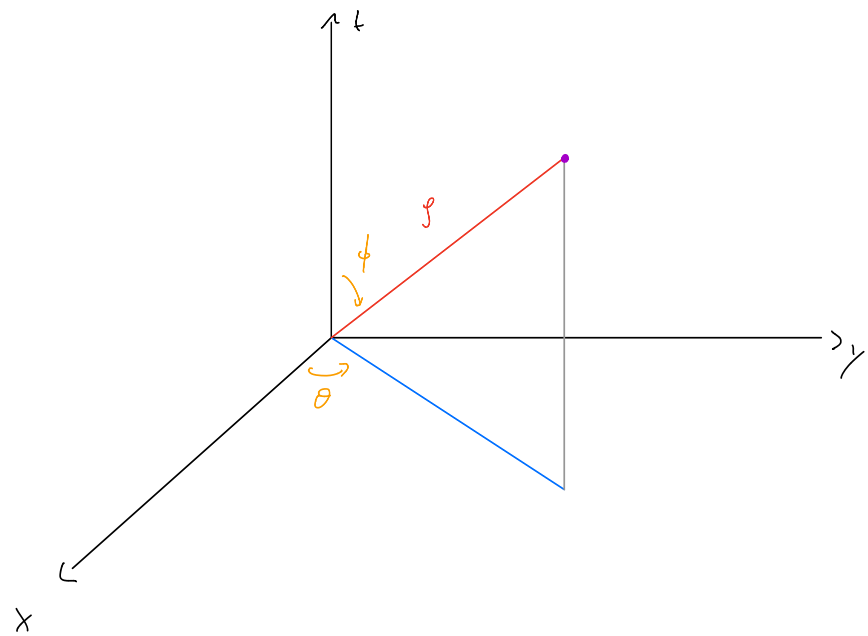
Ex: Sketch the region E given in cylindrical coordinates by

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 1 - r^2 \le z \le 4\}$$



Triple integrals in spherical coordinates

A point in $\mathbb{R}^3$ can be uniquely identified by its spherical coordinates (ρ, θ, ϕ) depicted below:



We have

$$0 \leq \rho < \infty, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

The relation to the Cartesian coordinates is

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \cos\theta, \quad z = \rho \cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Ex: Find the spherical coordinates of the point with Cartesian coordinates $(0, 2\sqrt{3}, -2)$.

$$\rho = \sqrt{0 + (2\sqrt{3})^2 + (-2)^2} = 4$$

$$\cos\phi = \frac{z}{\rho} = -\frac{2}{4} = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho \sin\phi} = 0 \Rightarrow \theta = \frac{\pi}{2}.$$

So $\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$ are the spherical coordinates.

Ex: Find the Cartesian coordinates of the point given in spherical coordinates by $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$.

$$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2 \cos\frac{\pi}{3} = 1.$$

So $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$ are the Cartesian coordinates.

If E is a region given in spherical coordinates by

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \phi \le \delta\}$$

then

$$\iiint_E f(x,y,z)\, dV$$

$$= \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi .$$

<u>Ex</u>: Let $E = \{x^2+y^2+y^2 \le 1\}$. Write

$\iiint_E f(x,y,z)\, dV$ as an integral in Cartesian coordinates and then in spherical coordinates, where $f(x,y,z) = xyz$.

In Cartesian:

$-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$,

$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$,

so

$$\iiint_E f(x,y,z)\,dV = \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$$

In spherical:

$$0 \le \rho \le 1,\quad 0 \le \theta \le 2\pi,\quad 0 \le \phi \le \pi,$$

so

$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta\,\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

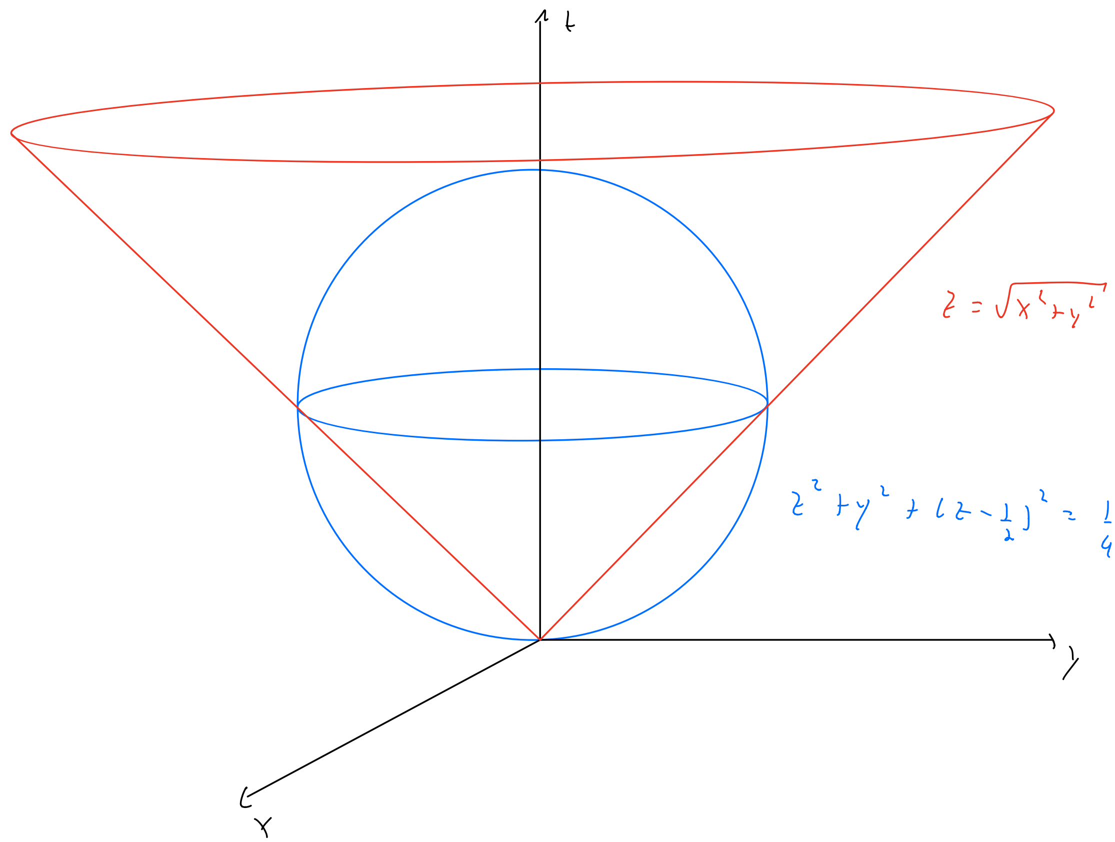
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^5 \sin^3\phi\cos\phi\cos\theta\sin\theta\,d\rho\,d\theta\,d\phi.$$

Ex: Write an iterated integral representing the volume of the solid between $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=z$.

Note that $x^2+y^2+z^2=z \iff x^2+y^2+z^2-z=0$

$$\iff x^2+y^2+\underbrace{z^2-2\cdot\tfrac{1}{2}z+\tfrac{1}{4}}_{=\left(z-\frac{1}{2}\right)^2}=\frac{1}{4}$$

$\Leftrightarrow \quad x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow$ sphere of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$.



In spherical coordinates:

sphere: $\quad x^2 + y^2 + z^2 = z$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi = \rho \cos\phi$$

$$\rho^2 \sin^2\phi + \rho^2 \cos^2\phi = \rho \cos\phi$$

$$\rho = \cos\phi$$

cone: $z = \sqrt{x^2 + y^2}$

$$\rho \cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$$

$$= \rho \sin\phi \quad (\text{note that } \sin\phi \geq 0)$$

So $\cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$, i.e. the equation of the cone in spherical coordinates is simply $\phi = \frac{\pi}{4}$.

The region is given by

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{4}, \quad 0 \leq \rho \leq \cos\phi.$$

Then

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

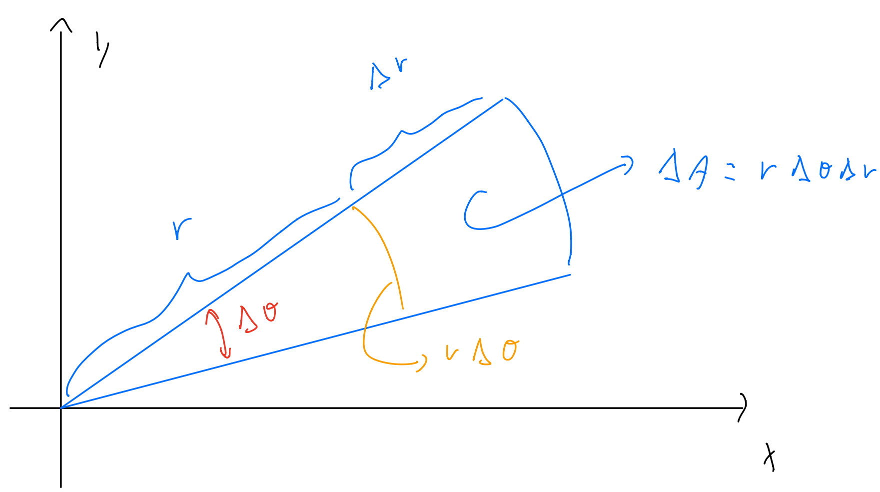
The volume element in cylindrical and spherical coordinates

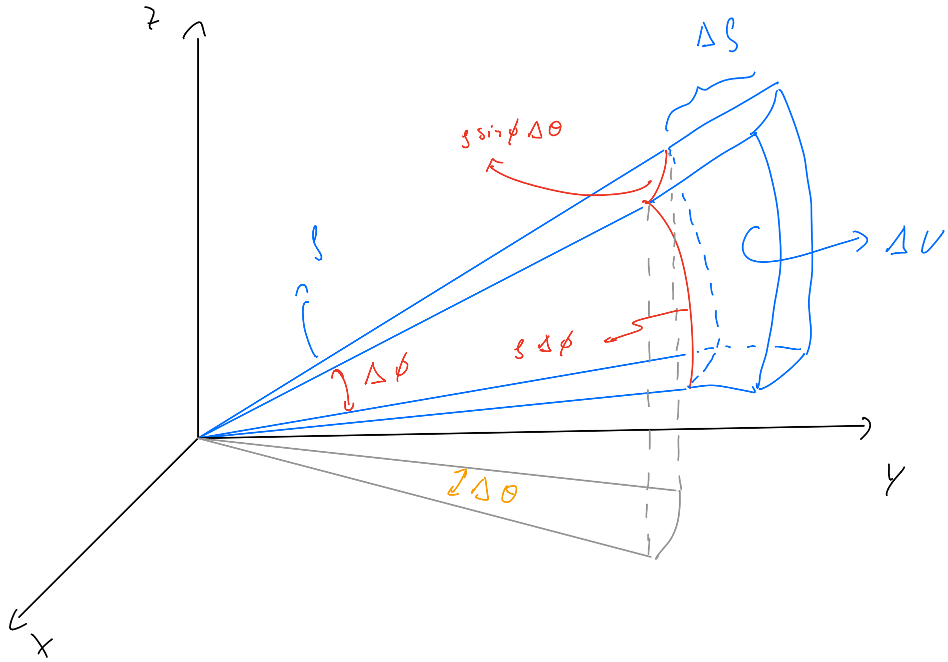
The integrals in cylindrical and spherical coordinates are, respectively

$$\iiint f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta,$$

$$\iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi,$$

with appropriate limits of integration. We can understand the factors r and $\rho^2\sin\phi$ from the pictures




z
y
x
$\Delta \rho$
$\rho \sin\phi \Delta\theta$
ρ
ΔV
$\rho \Delta\phi$
$\Delta\phi$
$\Delta\theta$

Change of variables in multiple integrals

In single-variable calculus, if we have

$$\int_a^b f(x)\,dx$$

and we make a change of variables $x = g(u)$, $a = g(c)$, $b = g(d)$, then the substitution rule gives

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du.$$

Since $x = g(u)$, $g'(u) = \frac{dx}{du}$ and we also write

$$= \int_c^d f(x(u))\,\frac{dx}{du}\,du.$$

Our goal is to generalize this formula for multiple integrals.

Consider a region S in $\mathbb{R}^2$ and a region R in $\mathbb{R}^2$. We distinguish these two regions by thinking that S is in the uv-plane and that R is in the xy-plane. A transformation between S and R is a function $T: S \to R$, which we write

$$T(u,v) = (x,y).$$

More explicitly

$$T(u,v) = (g(u,v), h(u,v)),$$

so we can write $x = g(u,v)$, $y = h(u,v)$. The transformation is called C^1 if g and h have continuous first order derivatives. T is called one-to-one if no two points in the domain of T have the same image, and onto if for any $(x,y) \in R$ is the image of at least one $(u,v) \in S$. If T is one-to-one and onto, then it has an inverse transformation $T^{-1}: R \to S$ that satisfies

$$T^{-1}(T(u,v)) = (u,v), \quad T(T^{-1}(x,y)) = (x,y).$$

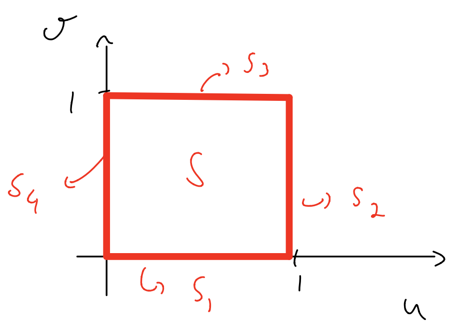
We can write

$$T^{-1}(x,y) = (G(x,y), H(x,y)).$$

Ex: Let

$$T(u,v) = (u^2 - v^2, 2uv),$$

i.e., $x = g(u,v) = u^2 - v^2$, $y = h(u,v) = 2uv$. Find the image of the square $S = [0,1] \times [0,1]$.



Let us find the effect of T on the boundary of S.

S_1: $0 \le u \le 1$, $v = 0$, $T(u,v) = (u^2, 0) = (x,y)$

S_2: $u = 1$, $0 \le v \le 1$, $T(u,v) = (1 - v^2, 2v) = (x,y)$

S_3: $0 \le u \le 1$, $v = 1$, $T(u,v) = (u^2 - 1, 2u) = (x,y)$

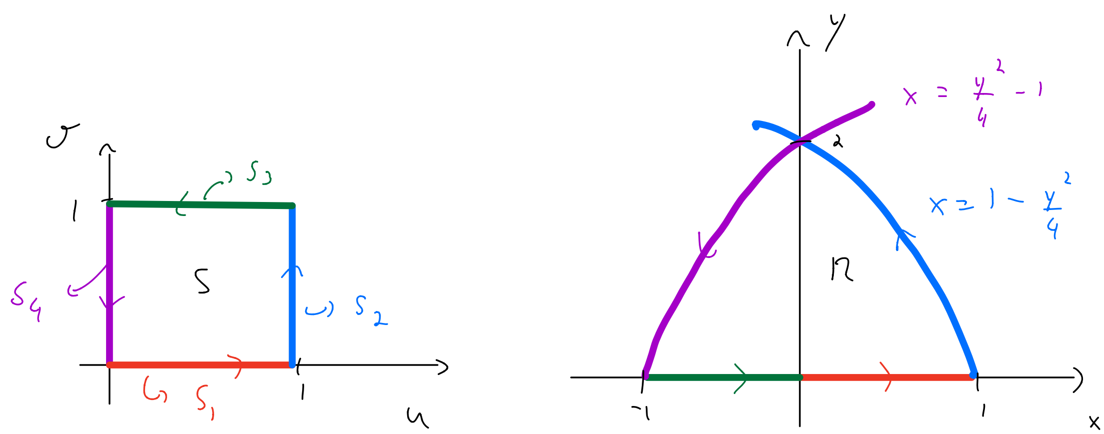
S_4: $u = 0$, $0 \le v \le 1$, $T(u,v) = (-v^2, 0) = (x,y)$.

S_1 is mapped into $[0,1] \times \{0\}$, S_4 into $[-1,0] \times \{0\}$.

For S_2, $v = \frac{y}{2}$, so $x = 1 - v^2 = 1 - \frac{y^2}{4}$.

For S_3, $u = \frac{y}{2}$, so $x = u^2 - 1 = \frac{y^2}{4} - 1$.

Thus



$R = T(S)$.

Def. The Jacobian of a transformation T given by $x = g(u,v)$, $y = h(u,v)$ is the determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Ex: For the transformation of the previous example, $x = g(u,v) = u^2 - v^2$, $y = h(u,v) = 2uv$, so

$$\partial_u x = 2u, \quad \partial_v x = -2v, \quad \partial_u y = 2v, \quad \partial_v y = 2u,$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \partial_u x \, \partial_v y - \partial_v x \, \partial_u y = 2u \cdot 2u - (-2v)2v$$
$$= 4u^2 + 4v^2.$$

Change of variable formula for double integrals. Suppose that $T : S \to R$ is C' transformation that maps S onto R and that T is one-to-one, except possibly on the boundary of S. Suppose that $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, that f is continuous on R, and that R is of type I or type II. Then

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Ex: Find $\iint_R y \, dA$, where R is the region in the first example above.

Since $T(S) = R$, we have

$$\iint_R y \, dA = \iint_S \underbrace{2uv}_{y} \underbrace{(4u^2+4v^2)}_{\frac{\partial(x,y)}{\partial(u,v)}} dA = 8\int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv = 2.$$

<u>EX</u>: Consider the change from polar coordinates to Cartesian coordinates

$$x = r\cos\theta \ , \quad y = r\sin\theta$$

so $x = x(r,\theta)$, $y = y(r,\theta)$. Then $\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial \theta} = -r\sin\theta$,

$\dfrac{\partial y}{\partial r} = \sin\theta$, $\dfrac{\partial y}{\partial \theta} = r\cos\theta$,

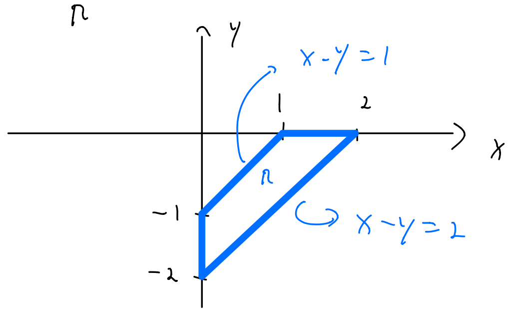
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$$

Thus

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,\overbrace{r}^{=\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}\,dr\,d\theta$$

which is another way to understand the factor r in $r\,dr\,d\theta$ in polar (and cylindrical) coordinates.

<u>EX</u>: Find $\displaystyle\iint_R e^{\frac{x+y}{x-y}}\,dA$ where R is the region:



This integral will be difficult to compute in xy variables, but if we change variables, we can simplify it. Set

$$u = x+y, \quad v = x-y.$$

Then

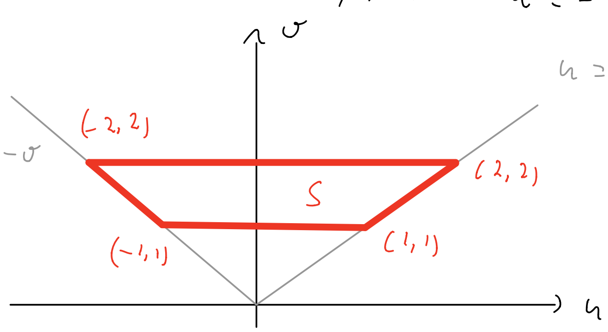
$$x = \frac{1}{2}(u+v), \quad y = \frac{1}{2}(u-v),$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = -\frac{1}{2}.$$

Let us find the region in the uv-plane corresponding to R. The lines $x-y=2$ and $x-y=1$ give $v=2$ and $v=1$. The line on the x-axis corresponds to $y=0$, $1 \le x \le 2$, so $u = x+0 = x$, $v = x-0 = x$, thus $u=v$, $1 \le v \le 2$.

The line on the y-axis corresponds to $x=0$, $-1 \le y \le -2$, so $u = 0+y = y$, $v = 0-y = -y$, thus $u=-v$, $1 \le v \le 2$.

Thus S is



thus $S = \{ 1 \le v \le 2, \ -v \le u \le v \}$.

thus

$$\iint_R e^{\frac{x+y}{x-y}} \, dA = \iint_S e^{\frac{u}{v}} \left| \overbrace{\frac{\partial(x,y)}{\partial(u,v)}}^{=1/2} \right| du\,dv$$

$$= \frac{1}{2} \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \, du\,dv = \frac{3}{4}(e - e^{-1}).$$

Triple integrals

If we have a change of variables

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w),$$

the Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

and

$$\iiint_R f(x,y,z) \, dA = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw.$$

$\underline{Ex}$: Consider the change from spherical to Cartesian coordinates

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

$$= \underline{-\rho^2\sin^3\phi\cos^2\theta} + \rho\sin\phi\sin\theta\underbrace{\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)}_{-\rho\sin\theta(\sin^2\phi+\cos^2\phi)}$$
$$\underline{-\rho^2\sin\phi\cos^2\phi\cos^2\theta}$$

$$= -\rho^2\sin\phi\cos^2\theta\left(\sin^2\phi + \cos^2\phi\right)$$

$$= -\rho^2\sin\phi\cos^2\theta - \rho^2\sin\phi\sin^2\theta = -\rho^2\sin\phi.$$ Thus

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\underbrace{\rho^2\sin\phi}_{=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|,\ \sin\phi>0}\,d\rho\,d\theta\,d\phi$$

giving the factor $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ for spherical coordinates.

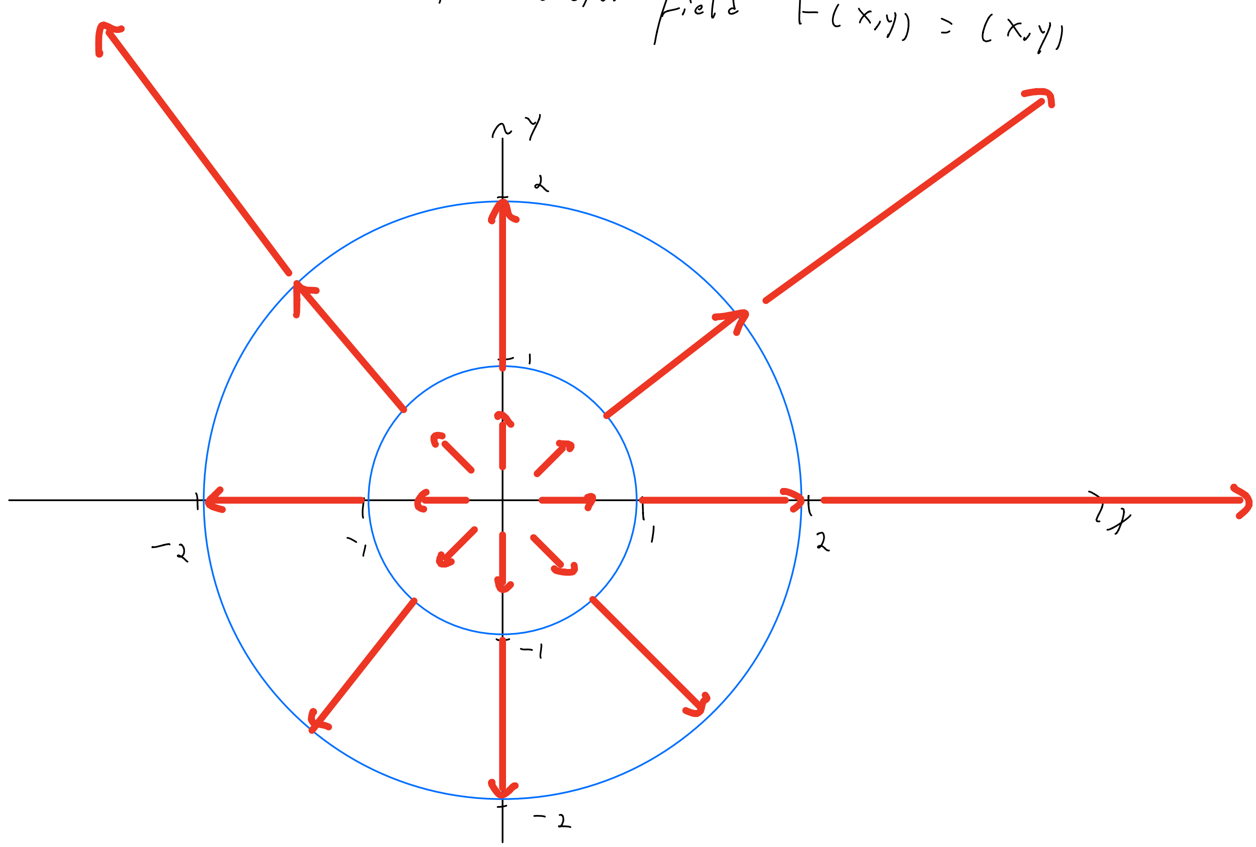
Vector fields

Def. Let $D \subset \mathbb{R}^2$. A vector field in D is a function $\vec{F}: D \to \mathbb{R}^2$. Similarly, if $D \subset \mathbb{R}^3$, a vector field in D is a function $\vec{F}: D \to \mathbb{R}^3$.

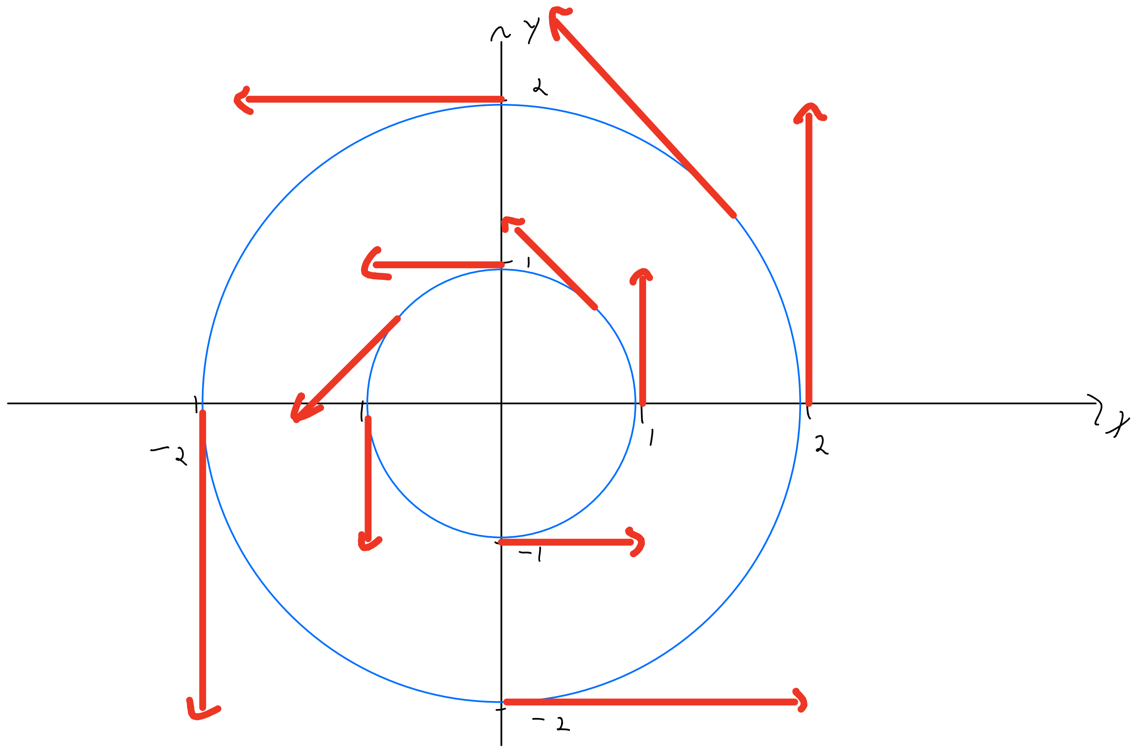
We can write a vector field as

$$\vec{F}(x,y) = p(x,y)\,\vec{i} + q(x,y)\,\vec{j} + r(x,y)\,\vec{k}.$$

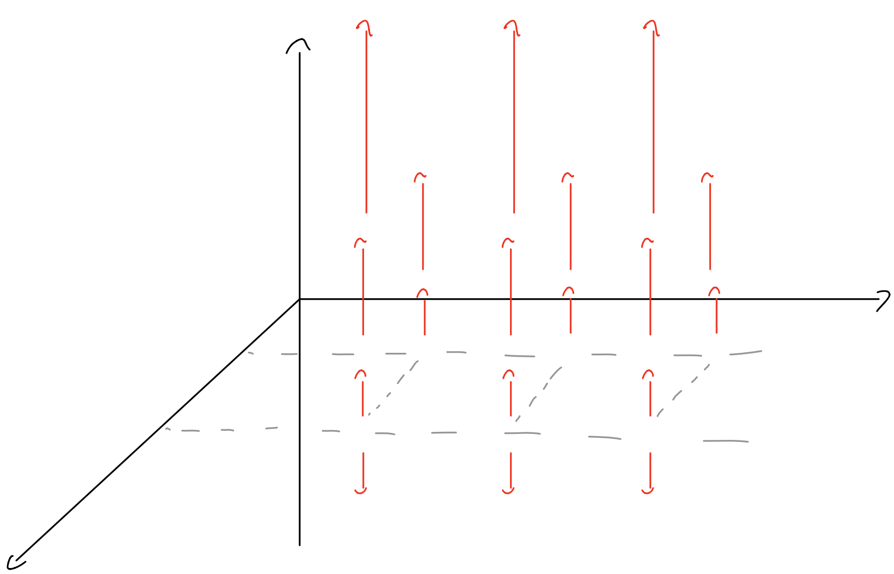
Ex: Sketch the vector field $\vec{F}(x,y) = (x,y)$



Ex: Sketch the vector field $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$



Ex: Sketch the vector field $\vec{F}(x,y,z) = z\vec{k}$



A vector field is called a gradient vector field or a conservative vector field if it is the gradient of a function:

$$\vec{F}(x,y,z) = \nabla f(x,y,z)$$
$$= \partial_x f(x,y,z)\,\vec{i} + \partial_y f(x,y,z)\,\vec{j} + \partial_z f(x,y,z)\,\vec{k}.$$

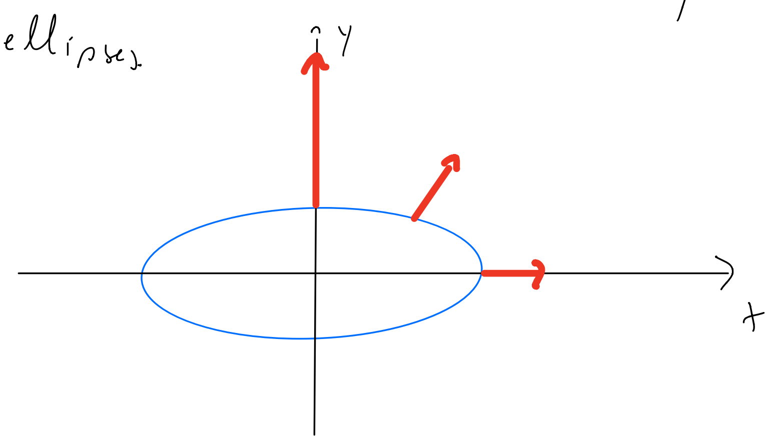
Ex: Sketch the gradient vector field for the function

$$f(x,y) = \frac{x^2}{2} + y^2 - 1.$$

$$\nabla f(x,y) = (x, 2y).$$

Consider the level curves $f(x,y) = \frac{x^2}{2} + y^2 - 1 = k$.

For any $k > -1$, they are ellipses, and ∇f is orthogonal to these ellipses.

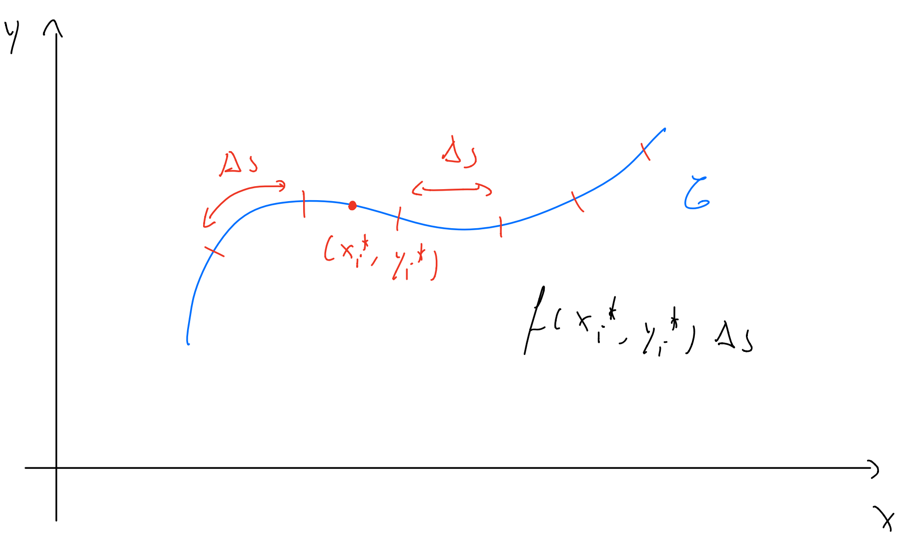


Line integrals

Suppose we have a curve $\mathcal{C}$ in the xy-plane and a function $f = f(x,y)$. The same way we compute the single variable integral of a function $h = h(x)$ by sampling among intervals $[x_{i-1}, x_i]$ of length Δx, i.e, we consider

$$\sum_{i=1}^{n} h(x_i^*)\,\Delta x \,,$$

we can imagine computing an integral of f by sampling along segments of length Δs on the curve $\mathcal{C}$



Def. Let $\mathcal{C}$ be a smooth curve given by $\vec{r}(t) = (x(t), y(t))$, $a \le t \le b$. Partion the interval $[a,b]$ into subintervals $[t_{i-1}, t_i]$ and let Δs_i be the length of the curve from $\vec{r}(t_{i-1})$ to $\vec{r}(t_i)$ and $(x_i^*, y_i^*) = \vec{r}(t_i^*)$, $t_i^* \in [t_{i-1}, t_i]$. The line integral of f along $\mathcal{C}$ is

$$\int_{\mathcal{C}} f(x,y)\, ds := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

provided the limit exists.

Recall that the arc-length function satisfies

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This implies that

$$\int_{\mathcal{C}} f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We sometimes call this integral the line integral with respect to arc-length. We can also define line integrals with respect to x and y by

$$\int_{\mathcal{C}} f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt,$$

$$\int_{\mathcal{C}} f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt.$$

All the standard properties of integrals hold for line integrals.

We can similarly define line integrals for curves in three dimensions:

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt.$$

Similarly,

$$\int_{\mathcal{C}} f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\,dt$$

$$\int_{\mathcal{C}} f(x,y,z)\,dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\,dt$$

$$\int_{\mathcal{C}} f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\,dt$$

Ex: Find $\int_{\mathcal{C}} y \sin z \, ds$ where $\mathcal{C}$ is given by $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$.

Compute

$x'(t) = -\sin t$,

$y'(t) = \cos t$,

$z'(t) = 1$,

So

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \sqrt{2}\, dt$$

Thus,

$$ds = \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$$

$$\int_{\mathcal{C}} y \sin z \, ds = \int_0^{2\pi} \underbrace{\sin t}_{=y} \underbrace{\sin t}_{=\sin z} \overbrace{\sqrt{2}\, dt}$$

$$= \sqrt{2} \int_0^{2\pi} \sin^2 t \, dt = \sqrt{2}\, \pi .$$

Def. Let $\vec{F}$ be a continuous vector field defined on a smooth curve $\mathcal{C}$ given by vector valued function $\vec{r}(t)$, $a \le t \le b$. The line integral of $\vec{F}$ along $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

Since we have $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$ and $ds = |\vec{r}\,'|\, dt$,

we can also write

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds.$$

Moreover, writing $\vec{F} = (P, Q, R)$, we have

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \big(P(\vec{r}(t)),\ Q(\vec{r}(t)),\ R(\vec{r}(t)) \big) \cdot \big(x'(t), y'(t), z'(t) \big)\, dt$$

$\vec{r}(t) = (x(t), y(t), z(t))$

$$= \int_a^b P(x(t), y(t), z(t))\, x'(t)\, dt$$

$$+ \int_a^b Q(x(t), y(t), z(t))\, y'(t)\, dt$$

$$+ \int_a^b R(x(t), y(t), z(t))\, z'(t)\, dt$$

$$= \int_C P\, dx + \int_C Q\, dy + \int_C R\, dz,$$

i.e.,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b P\, dx + \int_a^b Q\, dy + \int_a^b R\, dz,$$

which is another form of expressing the line integral of $\vec{F}$.

<u>Ex:</u> Find $\int_C \vec{F} \cdot d\vec{r}$ w/ $\vec{F}(x,y) = (x^2, -xy)$

and $C: (\cos t, \sin t),\ 0 \le t \le 2\pi$.

$$\vec{F}(\vec{r}(t)) = \vec{F}(\cos t, \sin t)$$

$$= (\cos^2 t, -\cos t \sin t)$$

$$\vec{r}'(t) = (-\sin t, \cos t)$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -2\cos^2 t \sin t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= -2\int_0^{2\pi} \cos^2 t \sin t\, dt$$

$$= -\frac{2}{3}.$$

The fundamental theorem of line integrals

The next theorem can be viewed as a version of the fundamental theorem of calculus,

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

for line integrals.

Theo. Let C be a smooth curve given by a vector valued function $\vec{r} = \vec{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

proof. Write

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b (f_x, f_y, f_z) \cdot (x', y', z')\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}\left(f(x(t), y(t), z(t)) \right) dt$$

$$= f(x(t), y(t), z(t)) \Big|_{t=a}^{t=b}$$

$$= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)). \qquad \square$$

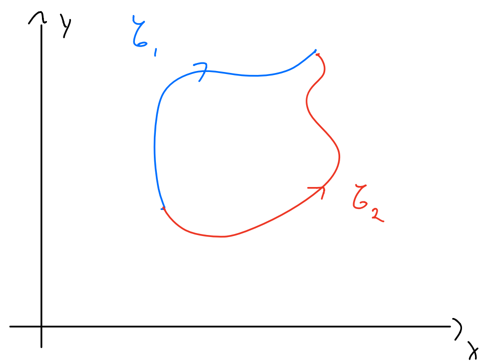
Ex: Compute $\int_C \nabla f \cdot d\vec{r}$ where C is the curve given by $\vec{r}(t) = (\cos 2t, \sin t)$, $0 \le t \le \pi$, and f is given by $f(x,y) = e^{xy}$.

$$\int_C \nabla f \cdot d\vec{r} = f(\cos 2t, \sin t) \Big|_0^{\pi} = f(1,0) - f(1,0) = 0$$

It follows from the theorem that if C is a a closed curve so that $\vec{r}(b) = \vec{r}(a)$, then

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = 0.$$

Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two piecewise smooth curves, which are called paths, that have the same initial and terminal points.



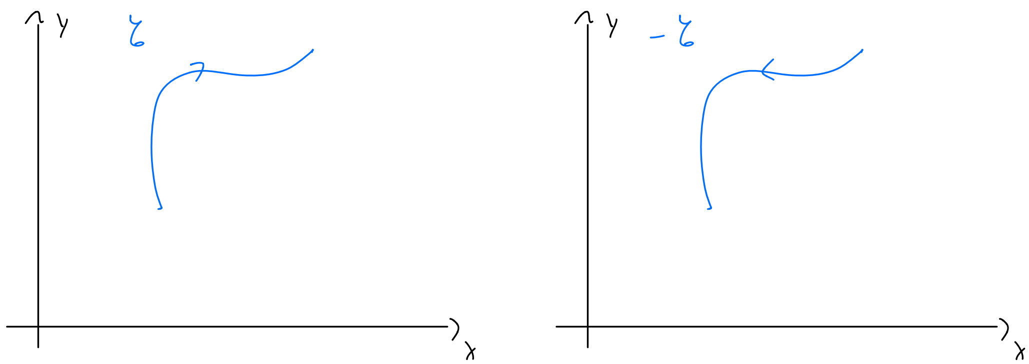
Then

$$\int_{\mathcal{C}_1} \nabla f \cdot d\vec{r} = \int_{\mathcal{C}_2} \nabla f \cdot d\vec{r}.$$

This is the case because if the curves are given by $\vec{r}_1$ and $\vec{r}_2$, then $\vec{r}_1(b) = \vec{r}_2(b)$ and $\vec{r}_1(a) = \vec{r}_2(a)$.

A vector field $\vec{F}$ is said to be independent of path if $\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}$ for any two curves with same initial and terminal points.

Give a curve $\mathcal{C}$, let us call by $-\mathcal{C}$ the curve obtained by reversing its direction, so that the initial and final points are switched.



If C is described by by $\vec{r}(t)$, then $\vec{R}(t) = \vec{r}(a+b-t)$ describes $-C$. Then

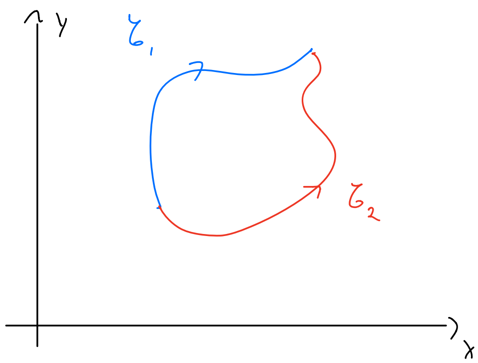
$$\int_{-C} \vec{F} \cdot d\vec{R} = \int_b^a \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt = \int_b^a -\vec{F}(\vec{r}(a+b-t)) \cdot \vec{r}'(a+b-t)\, dt$$

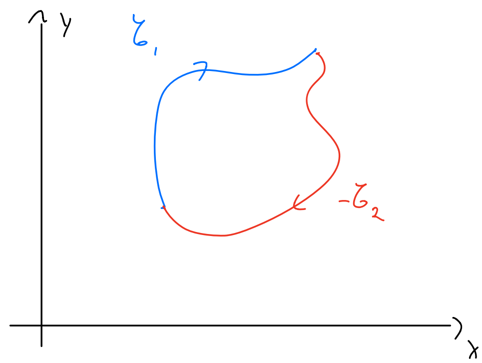
$$\underset{\tau = a+b-t,}{\downarrow} \quad = -\int_a^b \vec{F}(\vec{r}(\tau)) \cdot \vec{r}'(\tau)\, d\tau = -\int_C \vec{F} \cdot d\vec{r}$$

$$\vec{r}'(a+b-t) = \frac{d}{d\tau} \vec{r}(\tau)$$

Thus, if C_1 and C_2 have the same initial and terminal points, $\mathcal{C}_1 \cup (-\mathcal{C}_2)$ is a closed path and

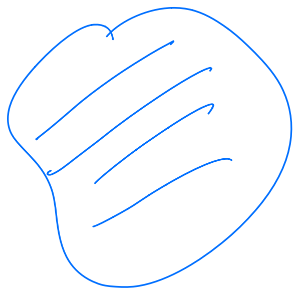
$$\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} \iff \int_{\mathcal{C}_1 \cup (-\mathcal{C}_2)} \vec{F} \cdot d\vec{r} = 0 .$$



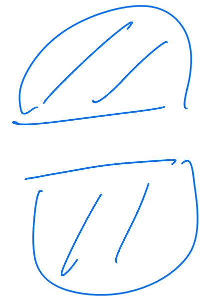


Thus we can say equivalently that $\vec{F}$ is independent of path if $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ for every closed curve $\mathcal{C}$.

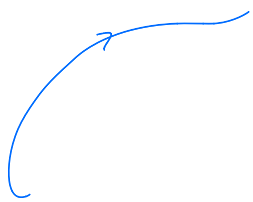
Def. A region $D \subset \mathbb{R}^2$ is called connected if any two points in D can be joined by a curve lying in D. A curve is called a simple curve if it does not intersect itself except at its initial and end points. $D \subset \mathbb{R}^2$ is called simply connected if every simple closed curve in D encloses only points of D (so D has no holes).



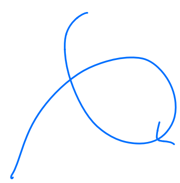
connected region



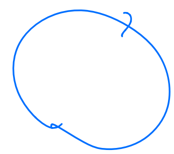
not connected region



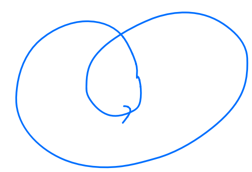
simple curve



not simple curve



simple closed curved



closed but not simple curve

connected but not simply connected region

simply connected region

Every conservative vector field is independent of path. The following theorem provides a converse.

<u>Theo</u>. Suppose that $\vec{F}$ is a continuous vector field on an open connected region. Assume that $\vec{F}$ is independent of path. Then $\vec{F} = \nabla f$ for some function f.

The following theorem is useful to determine whether a vector field is conservative:

<u>Theo</u>. The following statements hold:

- If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is conservative and P and Q have continuous partial derivatives in D, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

- If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$, P and Q have continuous partial derivatives in D, D is simply connected, and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\vec{F}$ is conservative.

Ex: Let $F(x,y) = (3+2xy,\ x^2 - 3y^2)$. If possible, find f such that $F = \nabla f$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(3+2xy) = 2x$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x$$

So $\partial_y P = \partial_x Q$. Since $\mathbb{R}^2$ is simply connected, f exists.

So $3 + 2xy = f_x$, $x^2 - 3y^2 = f_y$

$$\int f_x \, dx = 3x + x^2 y + g(y) = f(x,y)$$

$$f_y(x,y) = x^2 + g'(y) = x^2 - 3y^2 \Rightarrow g'(y) = -3y$$

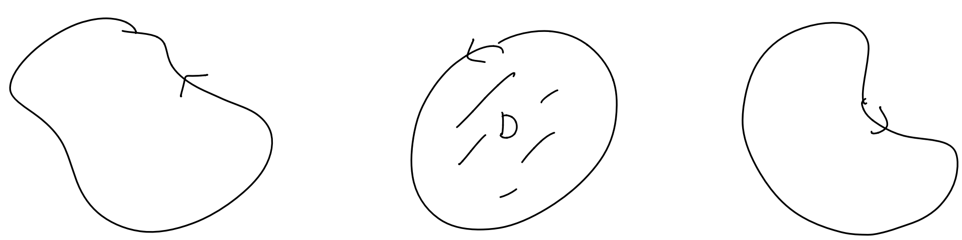
$$\Rightarrow g(y) = -y^3 + C$$

$$f(x,y) = 3x + x^2 y - y^3 + C.$$

Green's theorem

We will now see another type of generalization of the fundamental theorem of calculus, known as Green's theorem.

Def. A simple closed curve C is positively oriented (or has positive orientation) if it is traversed counterclockwise. If a region D is bounded by a curve C that is the union of simple closed curves, C is positively oriented if D is always "on the left" of the curve. The opposite orientation is called negative orientation.



positive orientation

positive orientation (D is on the left)

negative orientation

Note that we have defined orientation only for simple closed curves.

<u>Theo (Green's theorem)</u>. Let $\mathscr{C}$ be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$ and let D be the region bounded by $\mathscr{C}$. Let $P = P(x,y)$, $Q = Q(x,y)$ have continuous partial derivatives on an open region containing D. Then

$$\int_{\mathscr{C}} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

This is a generalization of the fundamental theorem of calculus in that it relates the integral of derivatives in the interior with the values of the function on the boundary.

Sometimes, we write

$$\oint_{\mathscr{C}} (P\,dx + Q\,dy) \quad \text{or} \quad \circlearrowleft\!\!\!\!\!\!\int_{\mathscr{C}} (P\,dx + Q\,dy)$$

to emphasize that the integral is over a closed curve positively oriented.

Ex: Find $\oint_{\mathcal{C}} (3y - e^{\sin x})\,dx + \oint_{\mathcal{C}} (7x + \sqrt{y^4+1})\,dy$,

where $\mathcal{C}$ is the circle $x^2+y^2=9$.

We have $P(x,y) = 3y - e^{\sin x}$, $Q(x,y) = 7x + \sqrt{y^4+1}$.

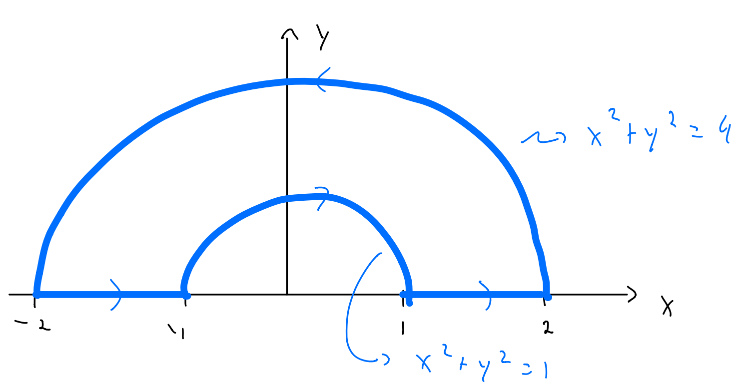
Applying Green's theorem, with $D = \{x^2+y^2 \leq 9\}$

$$\oint_{\mathcal{C}} (3y - e^{\sin x})\,dx + \oint_{\mathcal{C}} (7x + \sqrt{y^4+1})\,dy$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right) \right) dA$$

$$= \iint_D (7-3)\,dA = 4 \underbrace{\iint_D dA}_{=\pi 3^2} = 36\pi.$$

Ex: Find $\oint_{\mathcal{C}} (y^2\,dx + 3xy\,dy)$ where $\mathcal{C}$ is the curve



Let D be the region bounded by the curve.

Green's theorem gives:

$$\oint (y^2\,dx + 3xy\,dy) = \iint_D \left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right) dA$$

$$= \iint_D (3y - 2y)\,dA = \iint_D y\,dA.$$

The region D can be described in polar coordinates as

$$D = \left\{ (r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi \right\}.$$

Thus

$$\iint_D y\,dA = \int_0^{\pi} \int_1^2 r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr$$

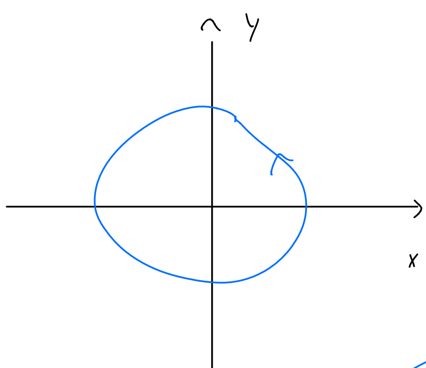
$$= -\cos\theta \Big|_0^{\pi} \cdot \frac{r^3}{3}\Big|_1^2$$

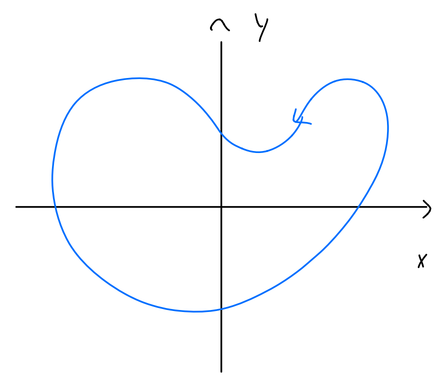
$$= \frac{14}{3}.$$

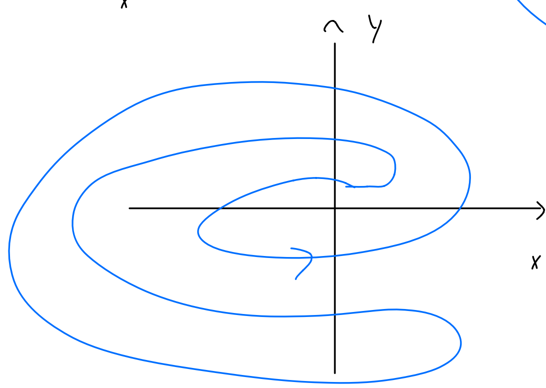
Ex: Let

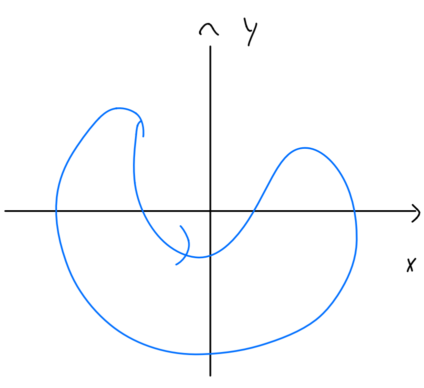
$$\vec{F}(x,y) = -\frac{y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}.$$

Find $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ where $\mathcal{C}$ is any simple closed curve with positive orientation and enclosing the origin. Examples of such curves are:









Not an example of a curve as in the statement

Compute

$$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{x^2+y^2}, \quad \frac{\partial P}{\partial y} = \frac{y^2-x^2}{x^2+y^2}$$

So

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Green's theorem would thus give $\oint_{\mathcal{C}} F \cdot d\vec{r} = 0.$

This, however, is wrong: We cannot apply Green's theorem directly because P and Q do not have continuous partial derivatives at $(0,0)$.

We can, however, apply Green's theorem for a curve $\tilde{\mathcal{C}}$ that does not contain the origin. In this case

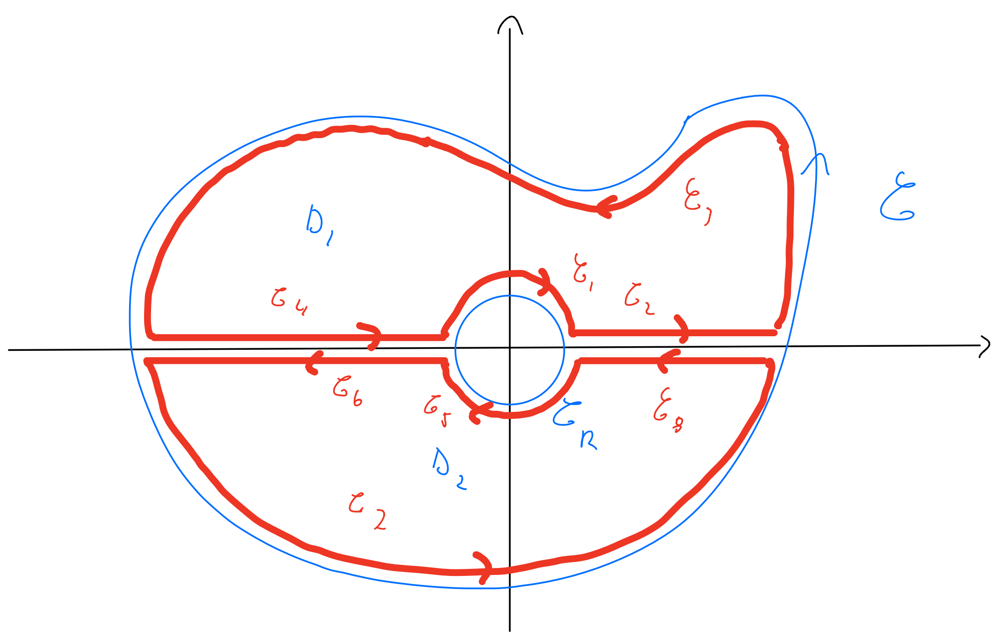
$$\oint_{\tilde{\mathcal{C}}} \vec{F} \cdot d\vec{r} = 0$$

Next, given a curve $\mathcal{C}$ enclosing the origin, let $\vec{r}(t) = (R\cos t, R\sin t)$ be such that the corresponding curve $\mathcal{C}_R$ is inside the region enclosed by $\mathcal{C}$. Then

$$\oint_{\mathcal{C}_R} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{\pi} \frac{-R\sin t\,(-R\sin t) + R\cos t\,R\cos t}{R^2\cos^2 t + R^2\sin^2 t}\,dt = 2\pi$$

Consider the curves in the picture



Then

$$\int_{C_4} \vec{F} \cdot d\vec{r} = -\int_{C_6} \vec{F} \cdot d\vec{r}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = -\int_{C_8} \vec{F} \cdot d\vec{r}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_5} \vec{F} \cdot d\vec{r} = -\int_{C_R} \vec{F} \cdot d\vec{r}$$

Thus

$$\int_{C} \vec{F} \cdot d\vec{r} = 2\pi .$$

Curl and divergence

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field in $\mathbb{R}^3$, its curl is the vector field in $\mathbb{R}^3$ defined by

$$\operatorname{curl}\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

We define the operator ∇ (named "nabla") by

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

Then

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}.$$

Notice that ∇ acting on a scalar function is the gradient:

$$\nabla f = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) f$$

$$= \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{k} + \frac{\partial f}{\partial z}\,\vec{k} .$$

<u>Ex</u>: Find $\nabla \times (2x,\ xy,\ xyz)$

$$\nabla \times (2x, xy, xyz) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & xy & xyz \end{bmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(xy) \right) - \vec{j}\left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(2x) \right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(2x) \right) = (xz, -yz, y).$$

Theo. If $f = f(x,y,z)$ has continuous second order partial derivatives then

$$\operatorname{curl}(\nabla f) = \vec{0}.$$

In particular, the curl of a conservative vector field is zero.

proof: Compute

$$\nabla \times (\nabla f) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \vec{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f^2}{\partial y \partial x}\right)$$

$= (0, 0, 0)$ by Clairaut's theorem. $\square$

Ex: Is it possible to find a function $f = f(x,y,z)$ such that $\nabla f = (2x, xy, xyz)$?

I.e., is $(2x, xy, xyz)$ conservative?

No. If that were the case, then $\text{curl}\,\nabla f = \vec{0}$, but $\text{curl}\,(2x, xy, xyz) \neq \vec{0}$.

If $\vec{F}$ is conservative, then $\text{curl}\,\vec{F} = \vec{0}$.
The next theorem is a converse.

Thm. If $\vec{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives and $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Ex: If possible, find f such that

$$\nabla f = (y^2 z^3,\ 2yxz^3,\ 3xy^2z^2).$$

Denote by $\vec{F}$ the RHS. Then

$$\nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2z^2 \end{bmatrix}$$

$$= \left(\frac{\partial}{\partial y}(3xy^2z^2) - \frac{\partial}{\partial z}(2xyz^3),\ -\frac{\partial}{\partial x}(3xy^2z^2) + \frac{\partial}{\partial z}(y^2z^3),\ \frac{\partial}{\partial x}(2xyz^3) - \frac{\partial}{\partial y}(y^2z^3) \right) = (0,0,0).$$

But the above theorem, there exists f such that $\nabla f = \vec{F}$. Thus

$$f_x(x,y,z) = y^2z^3 \Rightarrow f(x,y,z) = xy^2z^3 + g(y,z)$$

$$\Downarrow$$

$$f_y(x,y,z) = 2xyz^3 + g_y(y,z)$$

$$f_y(x,y,z) = 2xyz^3 \quad /\!/ \quad \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$$

$$\Downarrow$$

$$f_z(x,y,z) = 3xy^2z^2 + h'(z)$$

$$f_z(x,y,z) = 3xy^2z^2 \quad /\!/ \quad \Rightarrow h'(z) = 0 \Rightarrow h(z) = C.$$

$$f(x,y,z) = xy^2z^3 + C.$$

The divergence of $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is the scalar

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Using the dot product, we can write

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F},$$

for,

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(P\vec{i} + Q\vec{j} + R\vec{k}\right)$$

$$= \frac{\partial P}{\partial x}\vec{i}\cdot\vec{i} + \frac{\partial Q}{\partial y}\vec{j}\cdot\vec{j} + \frac{\partial R}{\partial z}\vec{k}\cdot\vec{k}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Ex: Find $\operatorname{div}(x, y, z)$.

$$\nabla \cdot (x, y, z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Thm. If $\vec{F} = (P, Q, R)$ is a vector field in $\mathbb{R}^3$ and P, Q, and R have continuous second order partial derivatives, then

$$\text{div curl } \vec{F} = 0.$$

proof.

div curl $\vec{F}$ =

$$\text{div}\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= 0.$$

$\square$

Ex: Let $\vec{F}(x, y, z) = (xz, xyz, -y^2)$.

Can we find a vector field $\vec{G}$ such that

$$\text{curl } \vec{G} = \vec{F} \text{ ?}$$

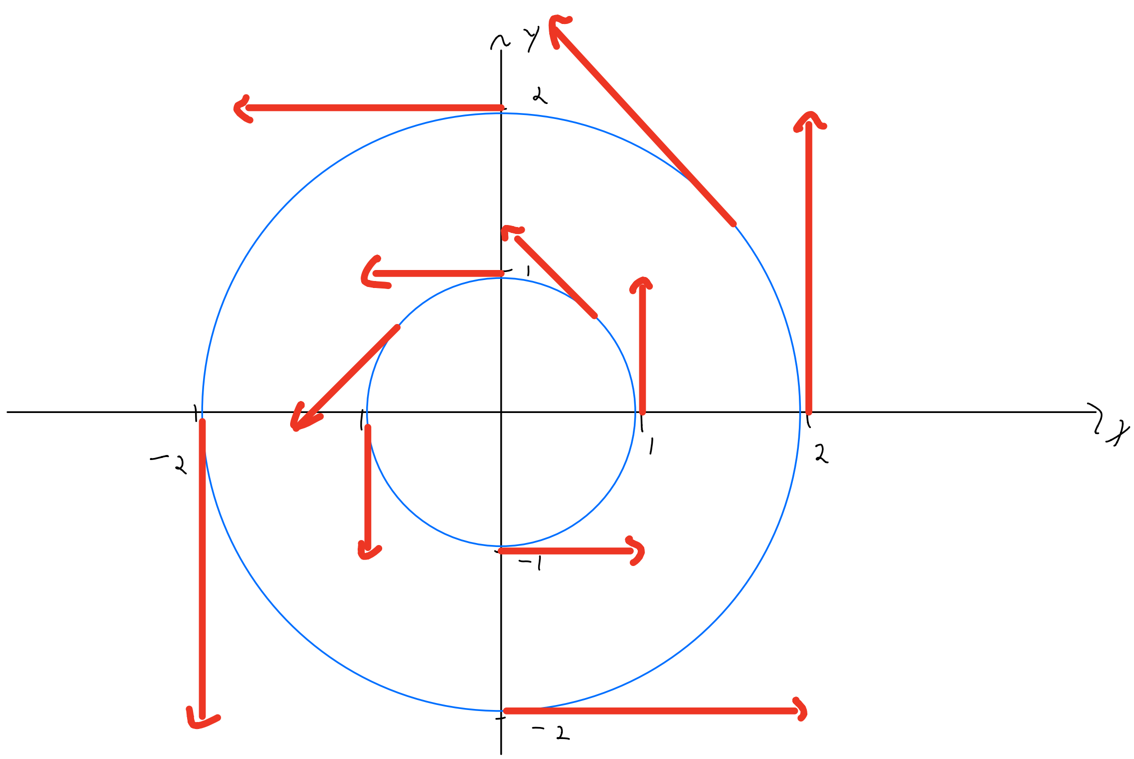
No. If that were the case,

$$\underbrace{\operatorname{div} \operatorname{curl} \vec{G}}_{=0} = \operatorname{div} \vec{F} = \operatorname{div}(xz, xyz, -y^2) = z + xz \neq 0$$

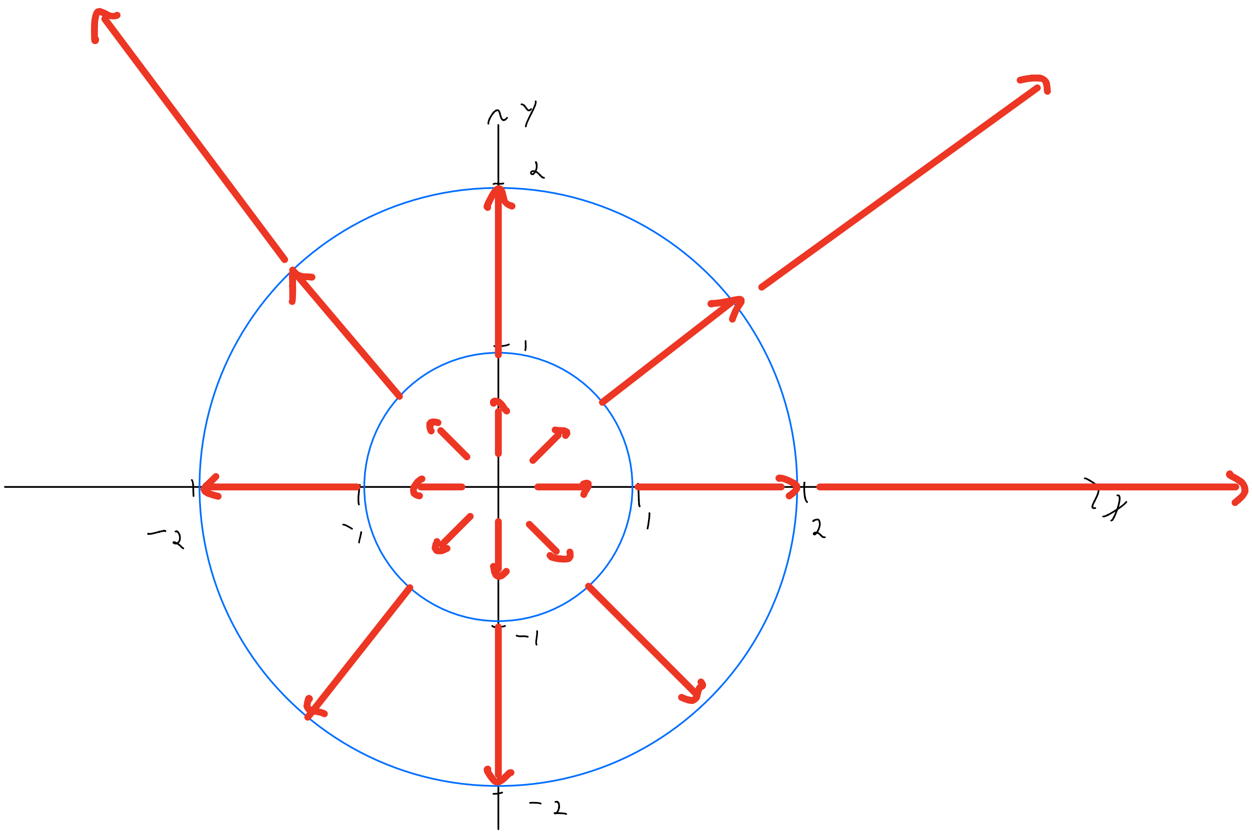
<u>Ex:</u> Use the vector fields

$\vec{F} = (-y, x, 0)$, $\vec{G} = (x, y, 0)$ to give a geometrical interpretation of curl and divergence.

$\vec{F}$:



$\vec{G}$:



$$\nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{bmatrix}$$

$$= \left(0,\ 0,\ \frac{\partial x}{\partial x} - \left(\frac{\partial(-y)}{\partial y}\right)\right) = (0, 0, 2)$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0$$

$$\nabla \times \vec{G} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{bmatrix} = (0, 0, 0)$$

$$\nabla \cdot \vec{G} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 2.$$

$\vec{F}$ rotates about the origin and $\vec{G}$ expands away from the origin. So curl measures the "rotation" of a vector field and div the "expansion" of a vector field.

Because of the above we say that $\vec{F}$ is irrotational if curl $\vec{F} = \vec{0}$ and incompressible if div $\vec{F} = 0$.

Using the curl and divergence we can restate Green's theorem in vector form as follows. We can view the vector field $\vec{F} = (P, Q)$ as a vector field in $\mathbb{R}^3$ by $\vec{F} = (P, Q, 0)$. Thus Green's theorem can be restated as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{F}) \cdot \vec{k} \, dA$$

$$\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_D \operatorname{div} \vec{F} \, dA$$

where $\vec{n}$ is the unit outward normal vector to C given by

$$\vec{n}(t) = \frac{x'(t)}{|\vec{r}\,'(t)|} \vec{i} - \frac{y'(t)}{|\vec{r}\,'(t)|} \vec{j}$$

if C is given by $\vec{r}(t) = (x(t), y(t))$, and ds is the arc length element.

An operator that appears often in physics and engineering is the <u>Laplace operator</u> denoted Δ or ∇^2:

$$\Delta f = \nabla^2 f = \operatorname{div}(\nabla f)$$

$$= f_{xx} + f_{yy} + f_{tt}.$$

Parametric surfaces

The same way we can describe a curve in $\mathbb{R}^3$ by a vector valued function $\vec{r}(t)$ of a single parameter t, we can describe a surface by a vector valued function $\vec{r}(u,v)$ of two parameters (u,v) defined in a region D of the uv-plane. More explicitly:

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}.$$

The image of $\vec{r}$, i.e., the set of all $(x,y,z) \in \mathbb{R}^3$ such that

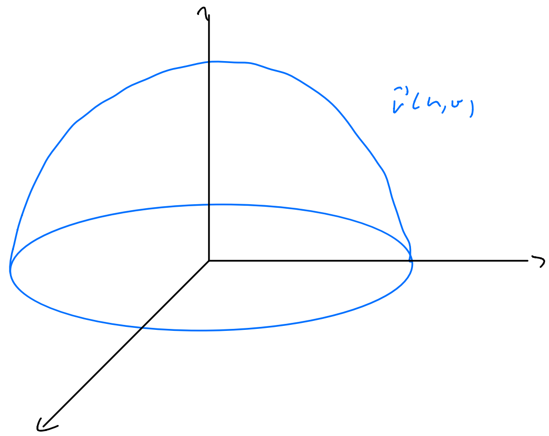
$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad (*)$$

$(u,v) \in D$, is called a <u>parametric surface</u> and equations $(*)$ are called the <u>parametric equations of the surface</u>.

<u>Ex</u>: The parametric surface

$$\vec{r}(u,v) = u\,\vec{i} + v\,\vec{j} + \sqrt{1-u^2-v^2}\,\vec{k},$$

$D = \{ (u,v) \mid u^2+v^2 \leq 1 \}$, is upper hemisphere of the sphere of radius one centered at $(0,0,0)$.



Ex: Sketch the parametric surface

$$\vec{r}(u,v) = 2\cos u\,\vec{i} + v\,\vec{j} + 2\sin u\,\vec{k}.$$

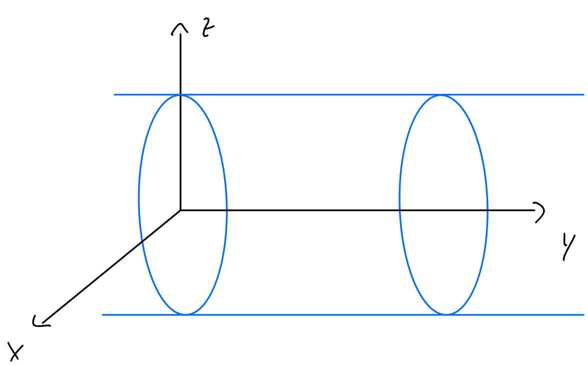
The parametric equations are

$$x = 2\cos u, \quad y = v, \quad z = 2\sin u.$$

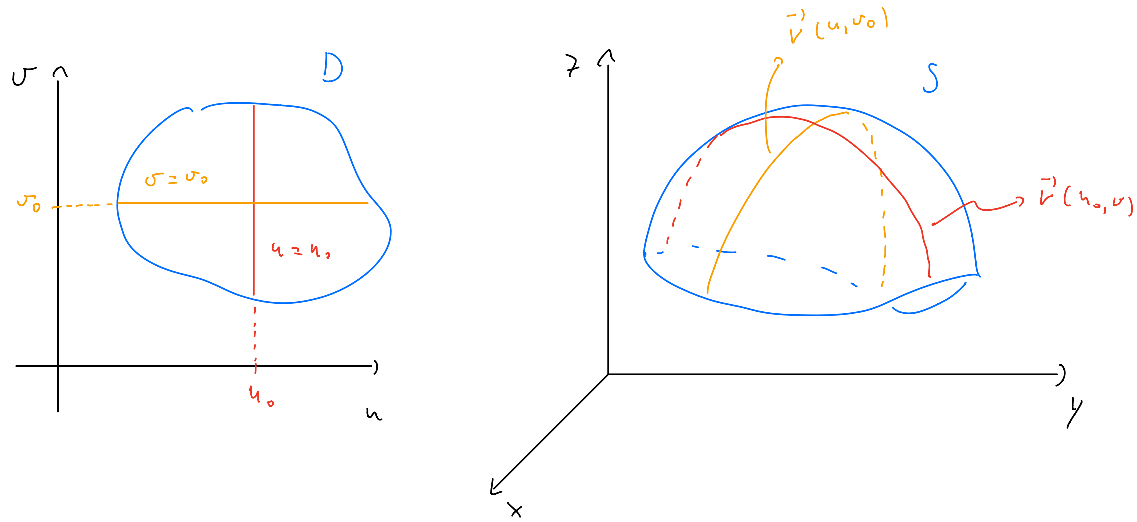
Thus $x^2 + z^2 = 4$. Since $y = v$ has no restrictions, the surface is given by

$$S = \{(x,y,z) \mid x^2+z^2=4, \ -\infty < y < \infty\}$$

which is a cylinder:



If a surface S is given by a vector valued function $\vec{r}(u,v)$, the curves of the form $\vec{r}(u_0, v)$ (curves of u constant) and $\vec{r}(u, v_0)$ (curves of v constant) are curves on S that form a grid called grid curves.



Ex: Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = R^2$$

and identify its grid curves.

Using spherical coordinates

$$x(\theta, \phi) = R \sin\phi \cos\theta, \quad y(\theta,\phi) = R\sin\phi \sin\theta, \quad z(\theta,\phi) = R\cos\phi$$

with $D = \{ (\theta, \phi) \mid 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi \}$, we have

$$\vec{r}(\theta,\phi) = R\sin\phi\cos\theta\,\vec{i} + R\sin\phi\sin\theta\,\vec{j} + R\cos\phi\,\vec{k}.$$

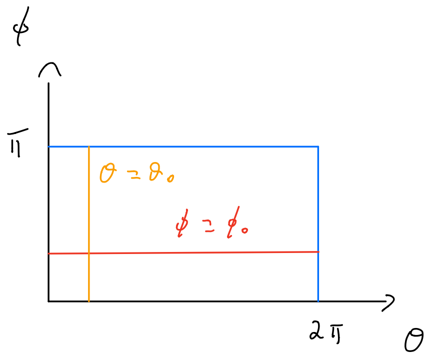
The line $\theta=\theta_0$ in D is mapped to

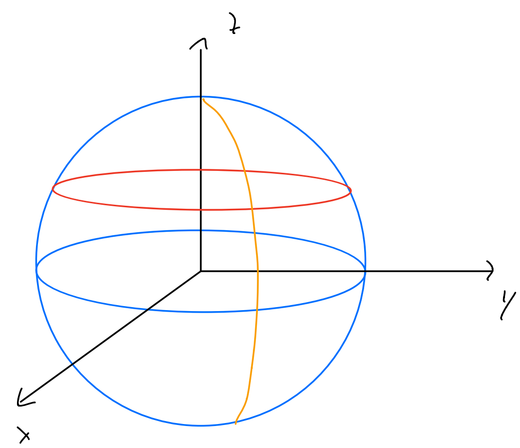
$$\vec{r}(\theta_0,\phi) = R\cos\phi\cos\theta_0\,\vec{i} + R\sin\phi\sin\theta_0\,\vec{j} + R\cos\phi\,\vec{k}$$

which is a meridian on the sphere, while the line $\phi=\phi_0$ in D is mapped to

$$\vec{r}(\theta,\phi_0) = R\cos\phi_0\cos\theta\,\vec{i} + R\sin\phi_0\sin\theta\,\vec{j} + R\cos\phi_0\,\vec{k}$$

which is a parallel on the sphere.

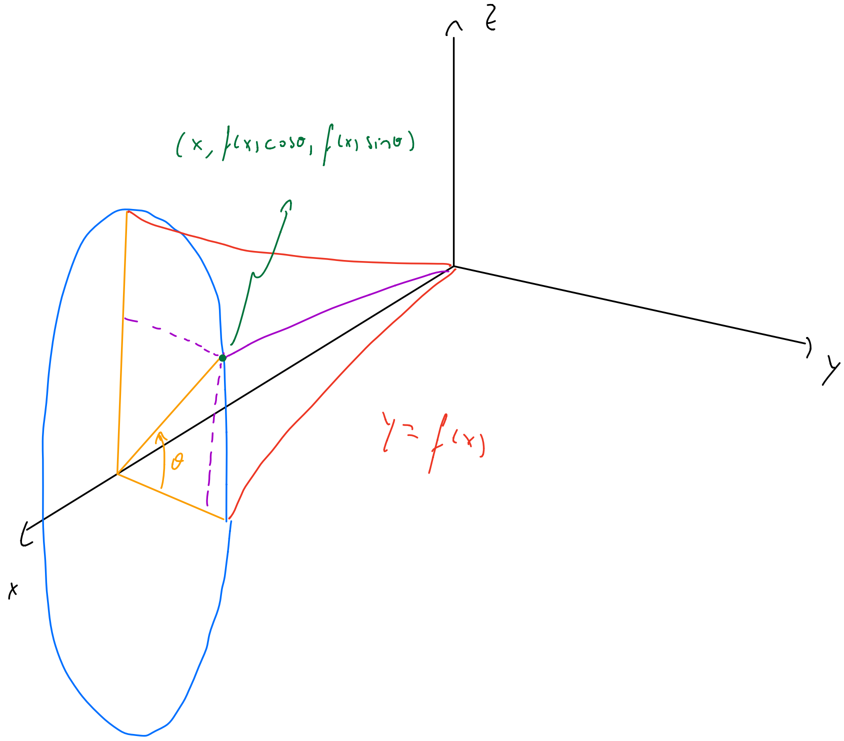




Surfaces of revolution, i.e., surfaces obtained by rotating a curve about an axis, can be described as parametric curves as follows.

Consider a curve $y = f(x)$, $f(x) \geq 0$, $a \leq x \leq b$. Rotating the curve by an angle θ we obtain

$$y = f(x)\cos\theta \quad , \quad z = f(x)\sin\theta$$



Thus

$$\vec{r}(x,\theta) = x\vec{i} + f(x)\cos\theta\,\vec{j} + f(x)\sin\theta\,\vec{k},$$

$a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.

Tangent planes

To find the tangent plane to a parametric surface S traced by

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k},$$

at $\vec{r}(u_0,v_0)$, consider the grid curves

$$\mathcal{C}_1 : \vec{r}(u, v_0), \qquad \mathcal{C}_2 : \vec{r}(u_0, v);$$

the vectors

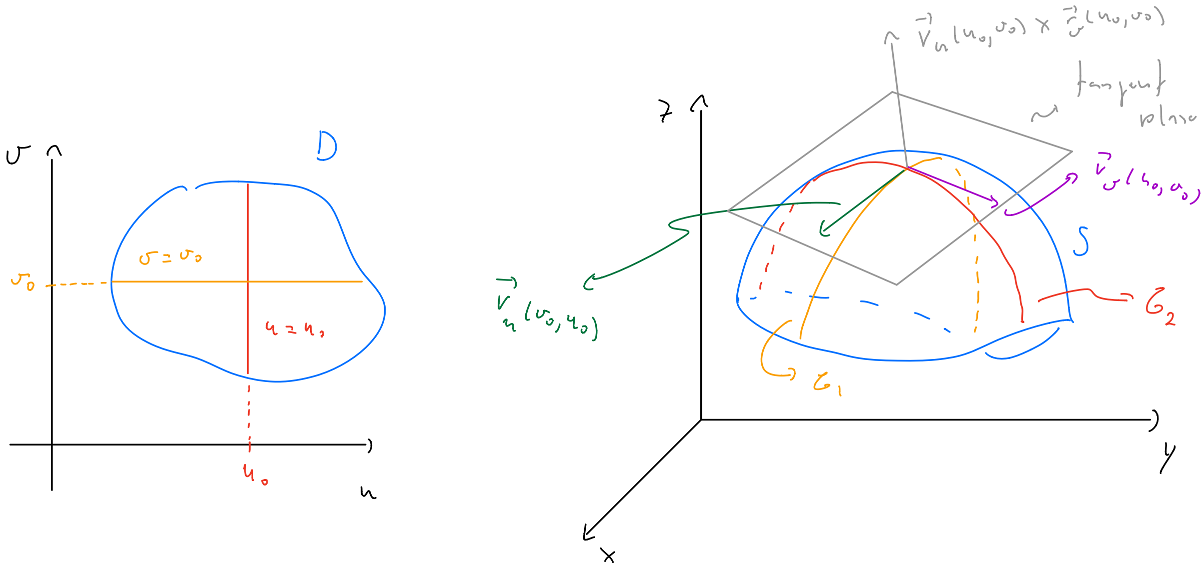
$$\frac{\partial \vec{r}}{\partial u}(u_0,v_0) = \vec{r}_u(u_0,v_0) = x_u(u_0,v_0)\,\vec{i} + y_u(u_0,v_0)\,\vec{j} + z_u(u_0,v_0)\,\vec{k}$$

and

$$\frac{\partial \vec{r}}{\partial v}(u_0,v_0) = \vec{r}_v(u_0,v_0) = x_v(u_0,v_0)\,\vec{i} + y_v(u_0,v_0)\,\vec{j} + z_v(u_0,v_0)\,\vec{k}$$

are tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. The surface S is called smooth (it has no corners) if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$.

In this case the tangent plane to S at $\vec{r}(u_0,v_0)$ is the plane containing $\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$. In this case $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$ is normal to the plane.



<u>Ex</u>: Find the tangent plane to

$$\vec{r}(u,v) = u^2\,\vec{i} + v^2\,\vec{j} + (u+2v)\,\vec{k}$$

at $(1,1,3)$.

$$\vec{r}_u(u,v) = 2u\,\vec{i} + \vec{k}$$

$$\vec{r}_v(u,v) = 2v\,\vec{j} + 2\vec{k}$$

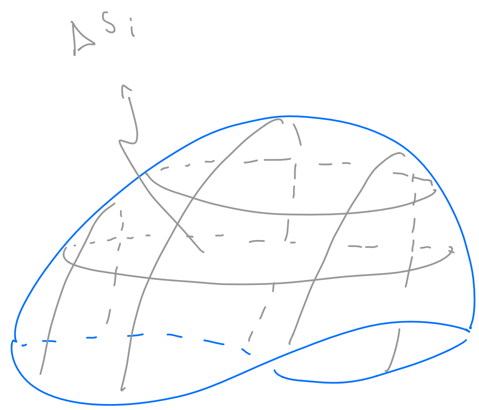
$\vec{r}_u \times \vec{r}_v = -2v\,\vec{i} - 4u\,\vec{j} + 4uv\,\vec{k}$. At $(1,1,3)$, $u=1, v=1$,

so $\vec{n} = (-2,-4,4)$ is a normal vector and the plane is

$$-2(x-1) - 4(y-1) + 4(z-3) = 0.$$

Surface area

If S is a smooth parametric surface, we can divide its surface area into small regions ΔS_i:



If ΔS_i is very small, its area is approximately $\Delta S_i \approx |\vec{r}_u \times \vec{r}_v|$. Passing to the limit:

$$\text{Area of } S = A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

where $D \subset \mathbb{R}^2$ is the domain of $\vec{r}$ (the integral is with respect to the (u,v) variables).

EX: Use the above formula to compute the area of a sphere of radius R.

We have

$$\vec{r}(\theta, \phi) = R \sin\phi \cos\theta \, \vec{i} + R \sin\phi \sin\theta \, \vec{j} + R \cos\phi \, \vec{k},$$

$(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$. Then

$$\vec{v}_\theta(\theta, \phi) = -R\sin\phi\sin\theta\,\vec{i} + R\sin\phi\cos\theta\,\vec{j}$$

$$\vec{v}_\phi(\theta, \phi) = R\cos\phi\cos\theta\,\vec{i} + R\cos\phi\sin\theta - R\sin\phi\,\vec{k}$$

$$\vec{v}_\theta \times \vec{v}_\phi = -R^2\sin^2\phi\cos\theta\,\vec{i} - R^2\sin^2\phi\sin\theta - R^2\sin\phi\cos\phi\,\vec{k}$$

$$|\vec{v}_\theta \times \vec{v}_\phi| = \sqrt{R^4\sin^4\phi\cos^2\theta + R^4\sin^4\phi\sin^2\theta + R^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{R^4\sin^4\phi + R^4\sin^2\phi\cos^2\phi} = \sqrt{R^4\sin^2\phi}$$

$$= R^2|\sin\phi| = R^2\sin\phi, \quad 0 \le \phi \le \pi.$$

Thus

$$A(S) = \iint_D |\vec{v}_\theta \times \vec{v}_\phi|\,dA = \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi\,d\phi\,d\theta$$

$$= 4\pi R^2.$$

If a surface is given as the graph of a function $z = f(x,y)$, then it can be written in parametric form as

$$\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + f(x,y)\,\vec{k}.$$

Then

$$\vec{r}_x = \vec{i} + f_x\,\vec{k}$$

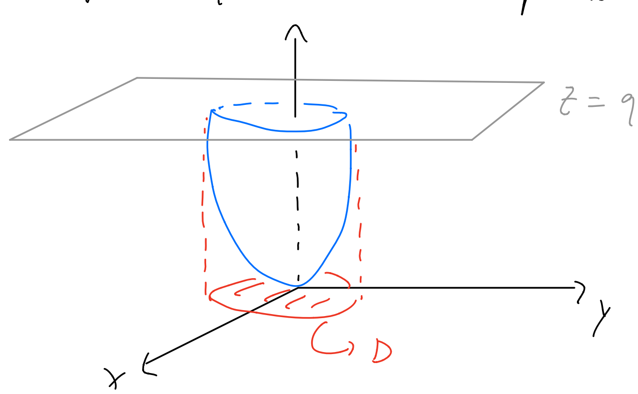
$$\vec{r}_y = \vec{j} + f_y\,\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{bmatrix} = -f_x\,\vec{i} - f_y\,\vec{j} + \vec{k}.$$

Thus

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\; dA.$$

Ex: Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



The values of (x, y) for which the graph is below $z = 9$ are $x^2 + y^2 \le 9$. Then

$$A(S) = \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dA = \iint_D \sqrt{1 + 4(x^2+y^2)}\, dA$$

Integrating in polar coordinates

$$= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\pi}{6}\left(37\sqrt{37} - 1\right).$$

If S is a surface of revolution obtained by rotating $y = f(x)$, $f(x) \ge 0$, $a \le x \le b$, about the x-axis, then, as seen above,

$$\vec{r}(x, \theta) = x\vec{i} + f(x)\cos\theta\, \vec{j} + f(x)\sin\theta\, \vec{k}, \quad a \le x \le b, \quad 0 \le \theta \le 2\pi.$$

Thus

$$\vec{r}_x = \vec{i} + f'(x)\cos\theta\, \vec{j} + f'(x)\sin\theta\, \vec{k}$$

$$\vec{r}_\theta = -f(x)\sin\theta\, \vec{j} + f(x)\cos\theta\, \vec{k}$$

$$\vec{r}_x \times \vec{r}_\theta = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{bmatrix}$$

$$= f(x)f'(x)\,\vec{i} - f(x)\cos\theta\,\vec{j} - f(x)\sin\theta\,\vec{k},$$

$$|\vec{r}_x \times \vec{r}_\theta| = \left((f(x))^2\left(1 + (f'(x))^2\right)\right)^{1/2}$$

$$= f(x)\sqrt{1 + (f'(x))^2},$$

thus

$$A = \int_0^{2\pi}\int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx\,d\theta$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

Surface integrals

We saw that the line integral

$$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\, dt$$

is a generalization of the single-variable integral

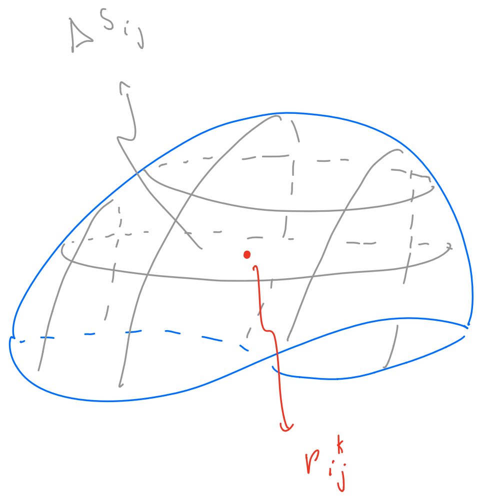
$$\int_a^b f(x)\, dx$$

where the domain of integration is no longer an interval but rather a curve in space (which is still one dimensional like the interval). Similarly, we can image integrals in two variables where the domain of integration is not a region in $\mathbb{R}^2$ but rather a surface S.

Using the (by now standard) idea of dividing S into small pieces, we can form the sum

$$\sum_{i=1}^{M} \sum_{j=1}^{N} f(P_{ij}^*)\, \Delta S_{ij},$$

where $P_{ij}^* \in \Delta S_{ij}$.



The surface integral of f over S is defined as

$$\iint_S f(x,y,z)\, dS = \lim_{N,M\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} f(P_{ij}^{*})\, \Delta S_{ij}.$$

Assume that S is a smooth surface given by $\vec{r}(u,v)$.

From the discussion of $A(S)$, we know that in the limit $M,N \to \infty$

$$\Delta S_{ij} \longrightarrow |\vec{r}_u \times \vec{r}_v|\, dA.$$

Thus,

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA.$$

$f(\vec{r}(u,v))$ means $f(x(u,v), y(u,v), z(u,v))$ since

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}.$$

Note that when $f(x,y,z) = 1$, we recover the formula for $A(S)$.

<u>Ex</u>: Find $\iint_S x^2\,dS$, where S is the unit sphere centered at the origin.

Using spherical coordinates, a parametric representation of the sphere is

$$\vec{r}(\theta,\phi) = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k},$$

$(\theta,\phi) \in [0,2\pi] \times [0,\pi]$.

$|\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi$ (see example above). Thus

$$\iint_S x^2\,dS = \iint_D (\sin\phi\cos\theta)^2 \sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\,\sin^3\phi\,d\phi\,d\theta = \frac{4\pi}{3}$$

If S is given by the graph of a function $z = g(x,y)$ then

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + g(x,y)\vec{k}.$$

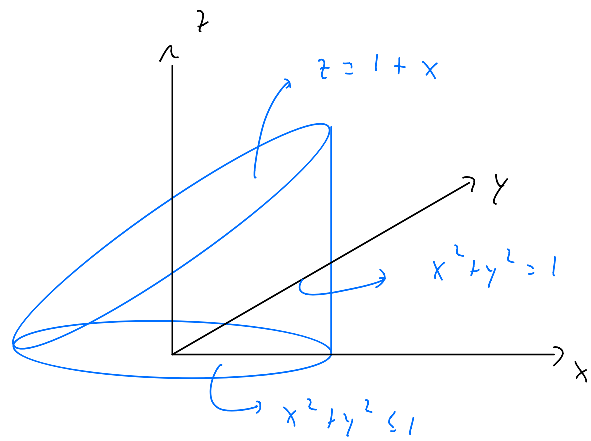
so that (see above discussion on graphs)

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{1 + (g_x(x,y))^2 + (g_y(x,y))^2}\, dA.$$

If S is a piece-wise smooth surface that is the union of smooth surfaces $S_1, \dots, S_n$, then we define

$$\iint_S f(x,y,z)\, dS = \iint_{S_1} f(x,y,z)\, dS + \cdots + \iint_{S_n} f(x,y,z)\, dS.$$

Ex: Find $\iint_S z\, dS$, where S is the surface whose sides are given by $x^2+y^2=1$, the top by the plane $z = 1+x$ lying above the bottom which is given by $x^2+y^2 \le 1$.



$S = S_1 \cup S_2 \cup S_3$, $\quad S_1 : x^2+y^2 = 1$, $S_2 : x^2+y^2 \le 1$, $S_3 : z = 1+x$

We can describe S_1 in cylindrical coordinates with

$$x = \cos\theta, \quad y = \sin\theta, \quad z = z,$$

$$0 \le \theta \le 2\pi, \quad 0 \le z \le 1 + x = 1 + \cos\theta$$

Thus

$$\vec{r}_1(\theta, z) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + z\,\vec{k}$$

$$(\vec{r}_1)_\theta(\theta, z) = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$(\vec{r}_1)_z(\theta, z) = \vec{k}$$

$$(\vec{r}_1)_\theta \times (\vec{r}_1)_z = \det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\left| (\vec{r}_1)_\theta \times (\vec{r}_1)_z \right| = 1$$

$$\iint_{S_1} z\,dS = \iint_D z\,|(\vec{r}_1)_\theta \times (\vec{r}_1)_z|\,dA$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta$$

$$= \frac{3\pi}{2}.$$

For S_2 we have $z=0$ so

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0.$$

For S_3, we have that it is the graph of $g(x,y) = 1+x$ for $x^2+y^2 \leq 1$, so

$$\iint_{S_3} z\,dS = \iint_D g(x,y)\sqrt{1+(g_x(x,y))^2+(g_y(x,y))^2}\,dA$$

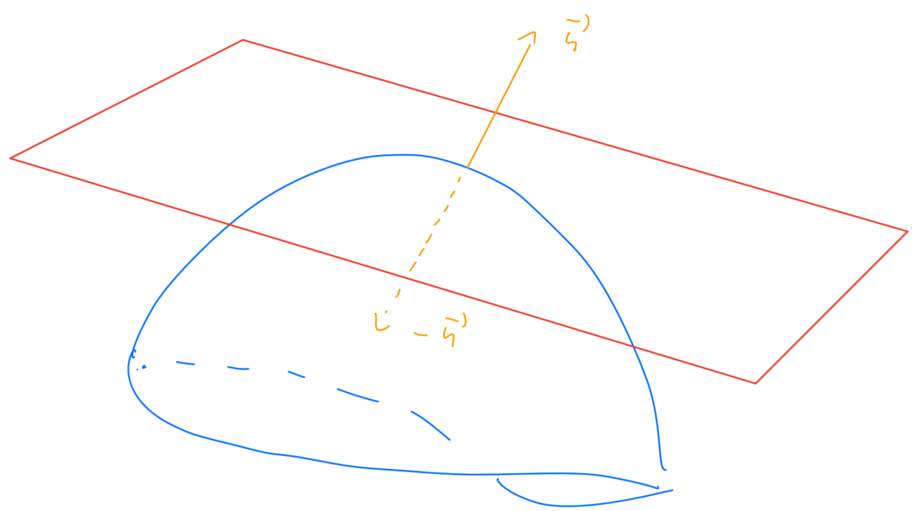
integrate in polar coordinates

$$= \iint_D (1+x)\sqrt{1+1^2+0^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{2}(1+r\cos\theta)\,r\,dr\,d\theta = \sqrt{2}\,\pi$$

Thus $$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS = \frac{3}{2}\pi + \sqrt{2}\,\pi.$$

Oriented surfaces

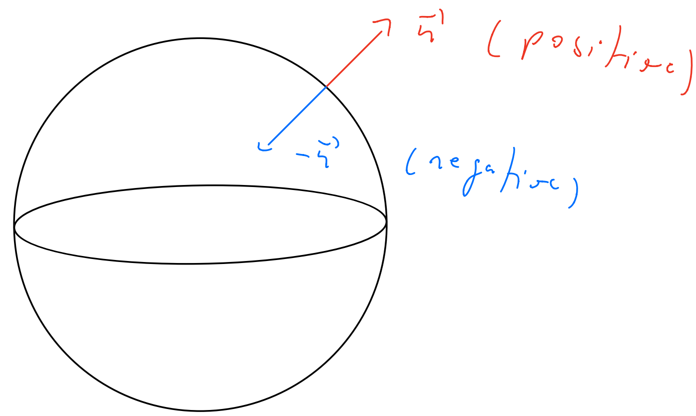
Consider a surface S that has a tangent plane at every point (except possibly at boundary points). There are two possible choices of unit normals for each tangent plane, $\vec{n}$ and $-\vec{n}$.



If it is possible to choose $\vec{n}$ continuously on S then the surface is called orientable, and non-orientable otherwise. An example of a non-orientable surface is the Möbius strip.

A choice of $\vec{n}$ over S for an orientable surface is called an orientation of S. A surface is called *oriented* if an orientation has been chosen.

For a closed (i.e., without boundary and bounded) orientable surface, we call the orientation where $\vec{n}$ points outward the *positive* orientation and the opposite orientation the *negative* orientation.



If S is a smooth orientable surface given in parametric form by $\vec{r}(u,v)$, then it is automatically given an orientation by

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

(with the opposite orientation given by $-\vec{n}$).

Ex: For the sphere of radius R centered at $(0,0,0)$,

$$\vec r(\theta,\phi) = R\sin\phi\cos\theta\,\vec i + R\sin\phi\cos\theta\,\vec j + R\cos\phi\,\vec k$$

$$\vec r_\theta \times \vec r_\phi = -R^2\sin^2\phi\cos\theta\,\vec i - R^2\sin^2\phi\sin\theta\,\vec j - R^2\sin\phi\cos\phi\,\vec k$$

$$|\vec r_\theta \times \vec r_\phi| = R^2\sin\phi,$$

$$\vec n = \frac{\vec r_\theta \times \vec r_\phi}{|\vec r_\theta \times \vec r_\phi|} = -\sin\phi\cos\theta\,\vec i - \sin\phi\sin\theta\,\vec j - \cos\phi\,\vec k = -\frac{1}{R}\vec r(\theta,\phi).$$

This $\vec n$ points inward, so the positive orientation is given by

$$-\vec n = \frac{1}{R}\vec r(\theta,\phi).$$

If S is a graph $z = g(x,y)$ then it also comes with an orientation given by

$$\vec n = \frac{-g_x\,\vec i - g_y\,\vec j + \vec k}{\sqrt{1 + (g_x)^2 + (g_y)^2}},$$

which is called the upward orientation ($-\vec n$ is called the downward orientation).

Surface integrals of vector fields

We have already learned how to integrate functions over surfaces. Now we will integrate vector fields over surfaces.

If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is defined as

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \vec{F} \cdot \vec{n} \, dS$$

which is also called the flux of $\vec{F}$ across S.

Observe that $\vec{F} \cdot \vec{n}$ is a function on S, so the RHS has already been defined.

If S is given by $\vec{r}(u,v)$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

so that

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, dS$$

Using the definition of the surface integral of a function

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

where D is the domain of $\vec{r}$.

<u>Ex</u>: Find the flux of $\vec{F}(x,y,z) = (z,y,x)$ across $x^2+y^2+z^2=1$. Use the positive orientation.

$$\vec{r}(\theta,\phi) = \sin\phi\cos\theta\,\vec{i} + \sin\phi\cos\theta\,\vec{j} + \cos\phi\,\vec{k},$$

$$\vec{r}_\theta \times \vec{r}_\phi = -\sin^2\phi\cos\theta\,\vec{i} - \sin^2\phi\sin\theta\,\vec{j} - \sin\phi\cos\phi\,\vec{k}$$

$$\vec{F}(\vec{r}(\theta,\phi)) = \cos\phi\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \sin\phi\cos\theta\,\vec{k}.$$

Thus (positive orientation)

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\theta,\phi))\cdot(-\vec{r}_\theta \times \vec{r}_\phi)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} (\cos\theta \sin^2\phi \cos\theta + \sin^3\phi \sin^2\theta + \sin^2\phi \cos\phi \cos\theta)\, d\phi\, d\theta$$

$$= \frac{4\pi}{3}$$

If S is given by a graph $z = g(x,y)$, then

$$\vec{F}\cdot(\vec{r}_x \times \vec{r}_y) = (P\vec{i} + Q\vec{j} + R\vec{k})\cdot(-g_x\vec{i} - g_y\vec{j} + \vec{k})$$

$$= -Pg_x - Qg_y + R$$

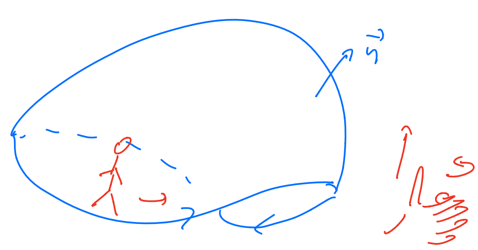
so

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\, dA.$$

Stokes' theorem

Green's theorem relates a (two-dimensional) integral in a region D in the xy-plane with a (one-dimensional) line integral over a curve that is the boundary of D. Stokes' theorem is a generalization of this idea, relating a (two-dimensional) integral over a surface S with a (one-dimensional) line integral over a curve that is the boundary of S.

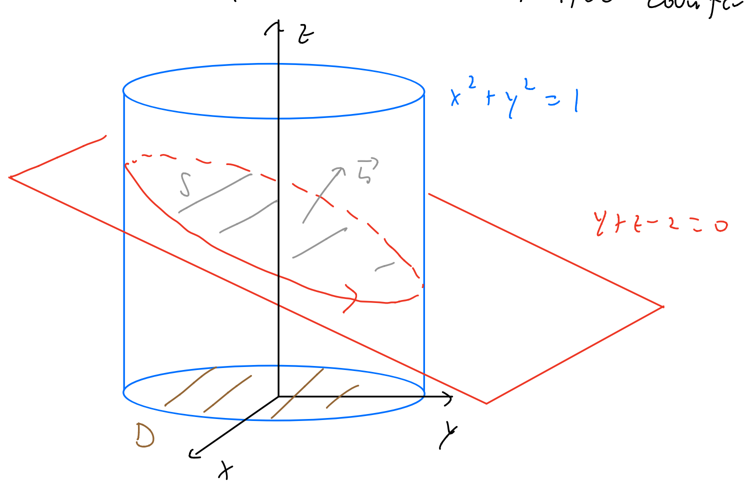
Let S be an oriented surface whose boundary is a curve $\mathcal{C}$. We say that the surface's orientation induces the positive orientation on $\mathcal{C}$ if $\mathcal{C}$ is oriented in such a way that if one "walks around" $\mathcal{C}$ with the "head" pointing in the same direction as the normal $\vec{n}$ to S (the normal that defines the orientation of S) then the curve is on the left.



Stokes' theorem. Let S be an oriented piece-wise smooth surface whose boundary is a simple, closed, piece-wise smooth curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives in an open region of $\mathbb{R}^3$ containing S. Then

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.$$

Ex: Use Stokes' theorem to find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$, and C is the intersection of the plane $y+z-2=0$ with the cylinder $x^2+y^2=1$ and whose projection on the xy-plane is oriented counterclockwise.



Compute

$$\operatorname{curl} \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{bmatrix} = (1+2y)\vec{k}$$

$\mathcal{C}$ bounds the surface S given by the graph $z = g(x,y) = 2-y$, with $x^2+y^2 \leq 1$ and normal pointing upward. Then

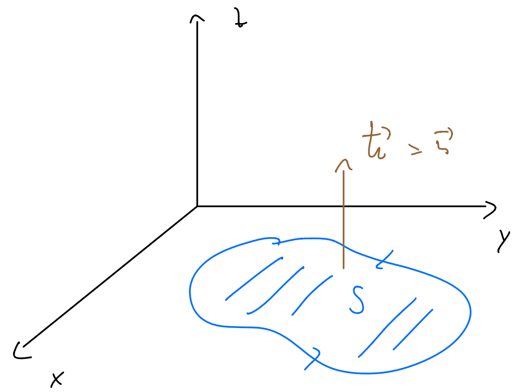
$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$= \iint_D (1+2y)\, dA,$$

where we used the formula for a surface integral when S is a graph and $D = \{(x,y) \mid x^2+y^2 \leq 1\}$. We can compute the integral in polar coordinates

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta = \pi.$$

Suppose that S lies on the xy-plane with upward orientation, so $\vec{n} = \vec{k}$. Then



$$\iint_S \operatorname{curl} \vec{F} \cdot dS$$

$$= \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS.$$

But since S lies on the xy-plane, $dS = dA$, and writing $S = D$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \cdot \vec{n}\, dA.$$

This is precisely Green's theorem in vector form. So, we see that Stokes' theorem is a generalization of Green's theorem.

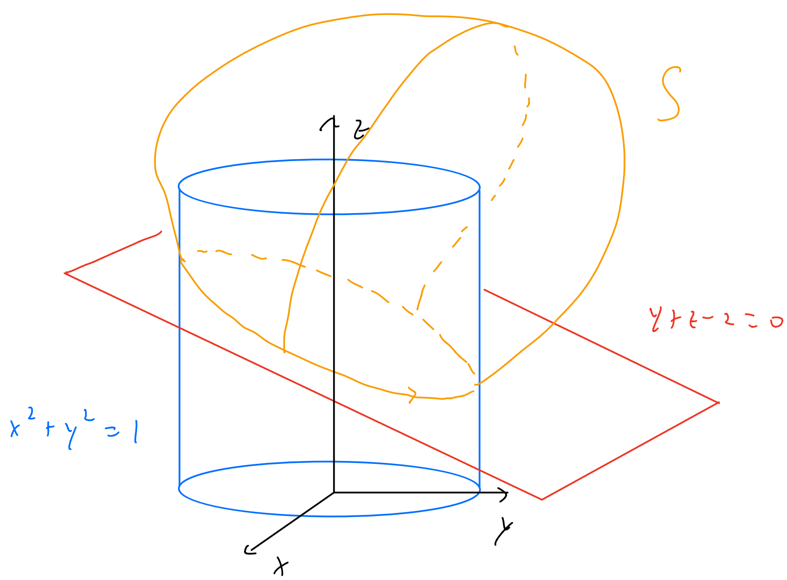
Suppose that we have two surfaces S_1 and S_2 with the same oriented boundary C. Then, by Stokes' theorem:

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{C} \vec{F} \cdot d\vec{r} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S},$$

i.e.,

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

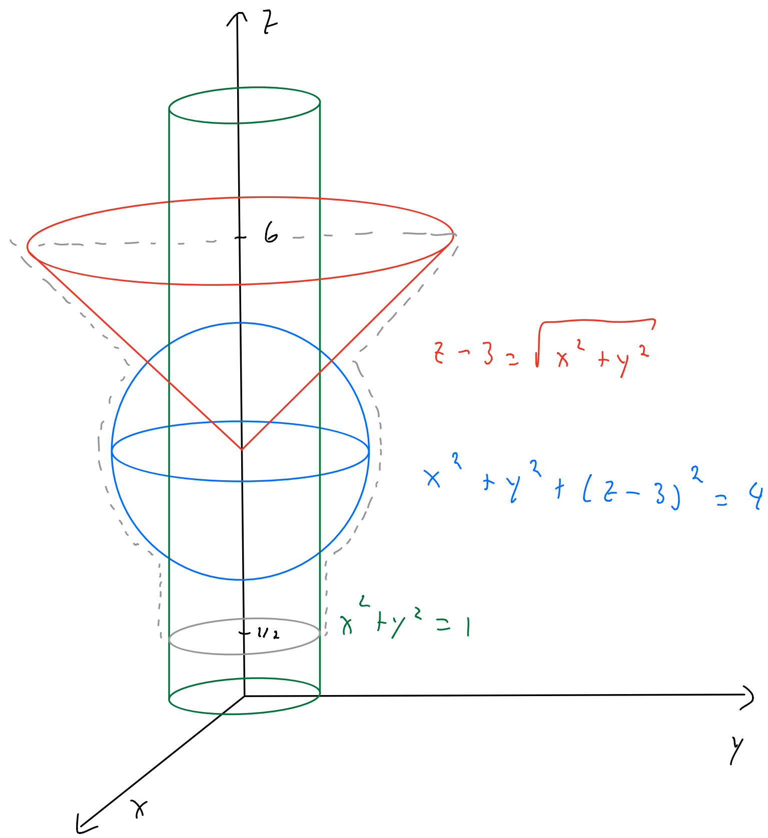
Ex: Find $\iint_{S} \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = -y^2 \vec{i} + x\vec{j} + z^2\vec{k}$ and S is the surface showed below:



Since $\iint_{S} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{C} \vec{F} \cdot d\vec{r}$ and the latter was computed in the previous example, $\iint_{S} \operatorname{curl} \vec{F} \cdot d\vec{S} = \pi$.

Ex: Find $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = (0, 0, z^2)$,

and S is the boundary of the largest region that is bounded by the sphere of radius two centered at $(0,0,3)$, $x^2+y^2=1$, $z-3=\sqrt{x^2+y^2}$, and $z=6$, and satisfying $z \geq \frac{1}{2}$, and $\vec{F} = \vec{k}$.



The surface is very complicated. Thus, instead of using a direct computation, let us use Stokes' theorem. For this, we need to write $\vec{F} = \operatorname{curl} \vec{G}$. All thing this is not always possible (recall $\operatorname{div} \operatorname{curl} \vec{u} = 0$), recalling our geometric interpretation of curl and divergence, we saw that

$$\operatorname{curl}(-y, x, 0) = (0, 0, 2),$$

so $\vec{G} = -\frac{1}{2} y \vec{i} + \frac{1}{2} x \vec{j}$ satisfies $\operatorname{curl} \vec{G} = \vec{k} = \vec{F}$. The boundary of S is the curve $\mathcal{C}$ given by $\vec{r} = (\cos t, \sin t, \frac{1}{2})$, $0 \le t \le 2\pi$. Here we are choosing the orientation such that $\mathcal{C}$ is oriented counterclockwise. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \operatorname{curl} \vec{G} \cdot d\vec{S} = \int_{\mathcal{C}} \vec{G} \cdot d\vec{r}.$$

Computing

$$\int_{\mathcal{C}} \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \frac{1}{2} \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \pi.$$

Ex: Use Stokes theorem to prove that if $\text{curl}\,\vec{F} = \vec{0}$ in $\mathbb{R}^3$ then $\vec{F}$ is conservative.

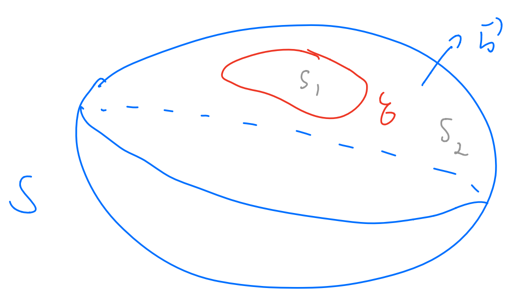
By the theorem:

$$\int_{C} \vec{F} \cdot d\vec{r} = \iint_{S} \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{S} \vec{0} \cdot d\vec{S} = 0.$$

Since C is any closed curve, this shows (by a previous result) that $\vec{F}$ is conservative.

Ex: Let S be a smooth closed surface. Show that $\iint_{S} \text{curl}\,\vec{F} \cdot d\vec{S} = 0.$

Let C be a simple closed curve contained in S. It split S in two surfaces S_1 and S_2:



Then

$$\iint_{S} \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{S_1} \text{curl}\,\vec{F} \cdot d\vec{S} + \iint_{S_2} \text{curl}\,\vec{F} \cdot d\vec{S}.$$

The orientation induced on $\mathcal{C}$ by S_1 is the opposite orientation of that induced by S_2, so:

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

$$\iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{-\mathcal{C}} \vec{F} \cdot d\vec{r} = -\int_{\mathcal{C}} \vec{F} \cdot d\vec{r},$$

giving the result.

The divergence theorem

We saw that we can write Green's theorem as

$$\int_{C} \vec{F} \cdot \vec{n} \, ds = \iint_{D} \operatorname{div} \vec{F} \, dA .$$

The divergence theorem generalizes this formula for three dimensions, where the double integral $\iint_{D}$ will become a triple integral and the (one-dimensional) line integral $\int_{C}$ will become a (two-dimensional) surface integral.

Recall that we defined regions of type 1, 2, and 3 when we talked about triple integrals. A region in $\mathbb{R}^3$ is called a simple solid region if it is simultaneously of type 1, 2, and 3.

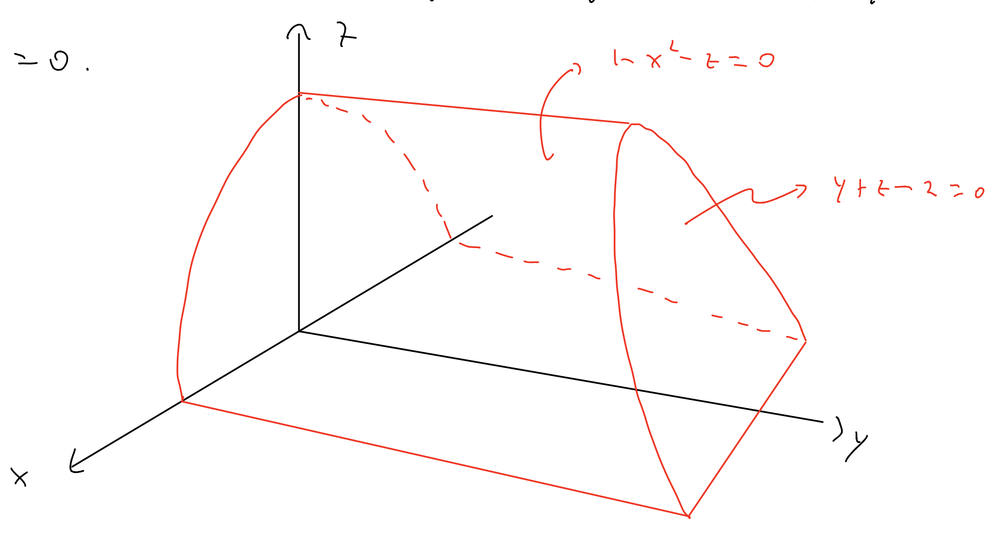
<u>The divergence theorem.</u> Let E be a simple solid region in $\mathbb{R}^3$ and let S be the surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing E. Then

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}.$$

<u>EX:</u> Find $\iint_S \vec{F} \cdot d\vec{S}$ where

$$\vec{F}(x,y,z) = xy\,\vec{i} + (y^2 + e^{xz^2})\,\vec{j} + \sin(xy)\,\vec{k},$$

and S is the surface of the region bounding $z=0$, $y=0$, $y+z-2=0$, $1-x^2-z=0$.



To evaluate the integral directly we have to split it in the four different surfaces composing the surface S. Let us use the divergence theorem:

$$\operatorname{div} \vec{F} = 3y.$$

We have

$$E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le z \le 1-x^2,\ 0 \le y \le 2-z\}.$$

thus

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3y\, dV$$

$$= 3\int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} y\, dy\, dz\, dx = \frac{184}{35}.$$

EX: Compute $\iint_S \vec{F} \cdot d\vec{S}$, where

$$\vec{F} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\vec{i} \quad \frac{y}{(x^2+y^2+z^2)^{3/2}}\vec{j}$$

$$\frac{z}{(x^2+y^2+z^2)^{3/2}}\vec{k},$$

and S is any closed surface containing the origin.

Compute

$$\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = -3\frac{x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{-3x^2 + x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}, \text{ and similarly}$$

$$\frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} = \frac{-3y^2 + x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}},$$

$$\frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{-3z^2 + x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}},$$

so

$$\operatorname{div} \vec{F} = \frac{-2x^2+y^2+z^2-2y^2+x^2+z^2-2z^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= 0.$$

We cannot apply the divergence theorem in the region containing the origin. Let S_r be the sphere of radius $r>0$ centered at the origin and consider $r>0$ small enough so that S_r is inside the region bounded by S. Then

$$\iint_{S_r} \vec{F} \cdot d\vec{S} = \iint_{S_r} \vec{F} \cdot \vec{n}\, dS = \iint_{S_r} \vec{F} \cdot \frac{(x,y,z)}{r}\, dS$$

$$= \iint_{S_r} \frac{(x,y,z)}{r^3} \cdot \frac{(x,y,z)}{r}\, dS = 4\pi.$$

Applying the divergence theorem in the

region between S and S_r

$$\iiint_E \operatorname{div} \vec{F}\, dV = 0 = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_r} \vec{F} \cdot d\vec{S}$$

$$= \iint_S \vec{F} \cdot d\vec{S} - 4\pi \text{ , so}$$

$$\iint_S \vec{F} \cdot d\vec{S} = 4\pi.$$