Recent developments in the theory of relativistic fluids

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Contents

1. Notation and convention

Unless stated otherwise, we adopt:

- Greek indices run from 0 to 3, Latin indices from 1 to 3, and repeated indices are summed over their range.
- $\{x^\alpha\}_{\alpha=0}^3$ denotes coordinates in spacetime, with $x^0 = t$ denoting a time coordinate and $\{x^i\}_{i=1}^3$ denoting spatial coordinates. We write $\{\frac{\partial}{\partial x^\alpha}\}_{\alpha=0}^3$ or simply $\{\partial_\alpha\}_{\alpha=0}^3$ for the corresponding basis of coordinate vectors.
- Signature convention for Lorentzian metrics is $- + ++$.
- Indices are raised and lowered with the spacetime metric.
- We use units where $C_l = 8\pi G = 1$, where C_l is the speed of light (in vacuum) and G is Newton's gravitational constant.
- H^N Denotes the Sobolev space with norm $\|\cdot\|_\mu$
- Def = definition, Theo = theorem, Prop = proposition, Ex = example

2. Introduction

The field of relativistic fluid dynamics is concerned with the study of fluids in situations when effects pertaining to the theory of relativity cannot be neglected. It is as essential tool in high-energy nuclear physics, cosmology, and astrophysics [20] [23]. Relativistic effects are manifest in models of relativistic fluids through the geometry of spacetime. This can be done in two ways: (a) by letting the fluid interact with a fixed spacetime geometry that is determined by a solution to vacuum Einstein's equations or (b) by considering the fluid equations coupled to Einstein's equations. In (a), we are neglecting the effects of the fluid's matter and energy on the curvature of spacetime, which in (b) such effects are taken into account. We will discuss both situations.

A crucial aspect of relativistic fluid dynamics is that the mathematical structures present in the equations od motions are substantially different than those present in classical (i.e., non-relativistic) fluids (e.g., the fluid velocity satisfies a constraint in the relativistic case, something with no analog in classical fluids). Thus, results for relativistic fluids cannot be obtained as a simple extension of techniques used for classical fluids.

3. Tools from Lorentzian geometry

The proper framework to discuss relativity and relativistic fluids is that of Lorentzian geometry. Since our goal is to get to fluids as soon as possible, we will only introduce some rudimentary notions that will be needed. Our approach is pragmatic in the sense that we will take the quickest route to the concepts we need, avoiding as much as possible of the discussion of the geometric structures involved. Students should be aware that by no means our discussion replaces an actual introduction to the topic, and that what follows does not necessarily consist of the most appropriate way of thinking about such concepts. Similar remarks apply throughout these notes whenever geometric concepts are needed. As introduction to Lorentzian geometry in the context of general relativity can be found in [15] and [22]. [BEE] and [O'u] offer an introduction to Lorentzian geometry as a topic on its own.

Remark 3.1. For simplicity, we introduce most of the concepts in $\mathbb{R}^4$. The generalization to differentiable manifolds is straightforward.

3.1. **Lorentzian metrics.**

Definition 3.2. A Lorentzian metric in $\mathbb{R}^4$ is a map that assigned to each $x \in \mathbb{R}^4$ a symmetric non-degenerate bilinear form $g(x) : \mathbb{R}^4 \to \mathbb{R}$ of signature $- + ++$. (Technical note: those familiar with geometry will notice that we identify $T_x\mathbb{R}^4$ with $\mathbb{R}^4$ itself; we will always make this identification) A spacetime is $\mathbb{R}^4$ endowed with a Lorentzian metric, $(\mathbb{R}^4, g)$.

Notation 3.3. We will often omit the x-dependence and write g for $g(x)$.

Thus, a Lorentzian metric is an inner product that is not positive-definite. Because of this we will often refer to $g(v, w), v, w \in \mathbb{R}^4$, as the (Lorentzian) inner product or simply product of v and w.

Notation 3.4. We will often say simply "metric" for a "Lorentzian metric."

Ex 3.5. The Minkowski metric is defined as follows.

Let $\tilde{m}$ be the matrix $\tilde{m} = \text{diagonal}(-1, 1, 1, 1)$. In standard rectangular coordinates we set

$$\begin{aligned} m(v, w) &= v^T \tilde{m} w = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3, \\ &= \tilde{m}_{\alpha\beta} v^\alpha v^\beta, \end{aligned}$$

where $v = (v^0, v^1, v^2, v^3)$, $w = (w^0, w^1, w^2, w^3) \in \mathbb{R}^4$. and $(\cdot)^T =$ transpose. $(\mathbb{R}^4, m)$ is the Minkowski space.

Note that m is a "constant" Lorentzian metric, i.e., it does not depend on $x \in \mathbb{R}^4$.

Of course, we can also express m with respect to other coordinates.

For example, taking (t, r, θ, ϕ), where (r, θ, ϕ) are spherical coordinates in $\mathbb{R}^3$, $\tilde{m}$ reads

$$\tilde{m} = \begin{bmatrix} m_{00} & & & \\ & m_{rr} & & \\ & & m_{\theta\theta} & \\ & & & m_{\phi\phi} \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{bmatrix}$$

where the entries not showed (e.g.,$m_{0r}, m_{r\theta}$,etc.) are zero.

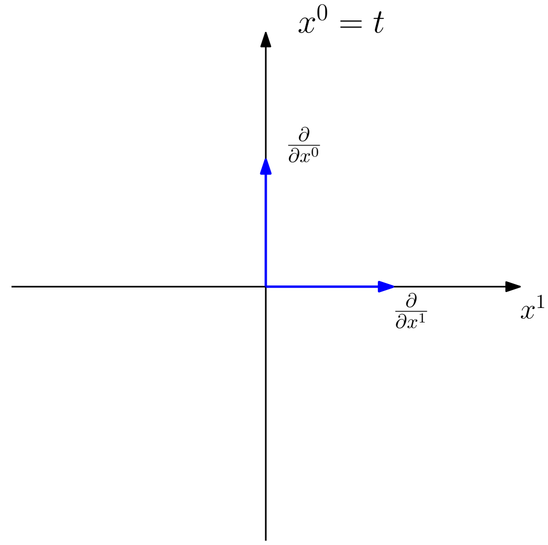
Ex 3.6. The Schwarzchild metric g_{sc} is defined by taking spherical coordinates (t, r, θ, ϕ) as in the previous example, and setting (for v and w expressed in spherical coordinates) $g_{sc}(v, w) = v^T \tilde{g}_{sc} w$, where (entries not shown are zero)

$$\tilde{g}_{sc} = \begin{bmatrix} -(1 - \frac{R}{r}) & & & \\ & (1 - \frac{R}{r})^{-1} & & \\ & & r^2 & \\ & & & r^2 \sin^2 \theta \end{bmatrix}$$, and R is a constant ($R = \frac{2GM}{c_l^2}$ is the usual presentation)

This expression is valid for $r > R$ only, but using different coordinates it can be extended to the whole of $\mathbb{R}^4$, see "Krushal extension."

Remark 3.7. For different choices of $v, w, m(v, w)$ can be $> 0, = 0, < 0$. Also, we can have $v \neq 0$ with $m(v, v) = 0$. Similar for g_{sc}. These are in fact general features of Lorentzian metrics.

More generally, consider a (Lorentzian) metric g and a coordinate basis $\{\frac{\partial}{\partial x^\alpha}\}_{\alpha=0}^3$. (In rectangular coordinates, $\{\frac{\partial}{\partial x^\alpha}\}_{\alpha=0}^3$ is just the canonical basis of $\mathbb{R}^4$. We fellow the standard notation of differential geometry. Recall our coordinate convention.) We define the matrix $\tilde{g}$ with entries $\tilde{g}_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$, Which is a symmetric matrix since g is a symmetric bilinear form. Then, $g(v, m) = v^T \tilde{g} w = \tilde{g}_{\alpha\beta} v^\alpha w^\beta$



Notation 3.8. From now on, we will write g for the matrix $\tilde{g}$ in a given basis, in practice identifying g with its matrix expression. Thus we write:

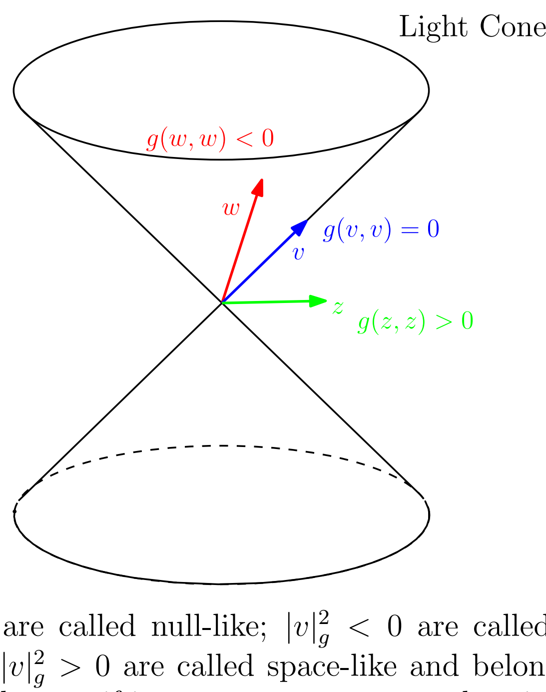
$$g(v, w) = g_{\alpha\beta} v^\alpha w^\beta.$$

The norm-squared(with respect to g) of a vector is defined by

$$|v|_g^2 = g(v, w) = g_{\alpha\beta} v^\alpha v^\beta$$

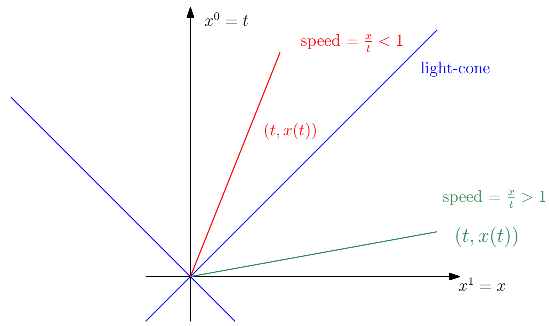
(Note that $|\cdot|_g$, sometimes written simply $|\cdot|$, is not really a norm).

A Lorentzian metric defines at each point $x \in \mathbb{R}^4$, a double cone called the light-cone by the set of vectors v based at x such that $|v|_g^2 = g(v, v) = 0$



Vectors such that $|v|_g^2 = 0$ are called null-like; $|v|_g^2 < 0$ are called time-like and belong to the "interior" of the light-cone; $|v|_g^2 > 0$ are called space-like and belong to the exterior of the light-cone. We call a curve time-like etc. if its tangent vector at each point is time-like etc.

In the theory of relativity, the light cones correspond to the region where light (i.e., electromagnetic radiation) propagates. Objects with mass propagate along time-like curves. No particle or information can propagate along space-like curves: this would mean that their speed is greater than the speed of light, violating a fundamental postulate of the theory of relativity (that "nothing propagates faster than the speed of light"). In Minkowski space, the picture is (recall our units convention):



Notation 3.9. Since g is non-degenerate (i.e., $g(v, w) = 0$ for all w implies $v = 0$), the matrix $(g_{\alpha\beta})$ is invertible. We denote the entries of the inverse matrix by $g^{\alpha\beta}$. Thus

$$g^{\alpha\beta} g_{\beta r} = \delta^\alpha_r$$

where δ^α_r is the Kronecker Delta

3.2. **Covariant derivative.** A concept that will be important for us is that of a directional derivative, i.e., derivative is the direction of a vector X. Conceptually, this involves "projecting" onto X. Because of this projection, the directional derivative will depend on the inner-product g. In multivariable calculus, we define the derivative in the direction of X by

$$\nabla_X = X \cdot \nabla$$

where $\cdot$ is the Euclidean inner product.∇_X acts on a scalar function f by $\nabla_X f = X \cdot \nabla f = X^\alpha \partial_\alpha f$, and on a vector field v componentwise, i.e., $(\nabla_X v)^\alpha = X^\beta \partial_\beta v^\alpha$. Moreover,the product rule holds,i.e.,$\nabla_X (v \cdot w) = (\nabla_X v) \cdot w + v \cdot (\nabla_X w)$. Note the manifest dependence of ∇_X on the Euclidean inner product.

We want something similar when the inner product is given by a metric g.

Definition 3.10. The covariant derivative of a vector field v in the direction of X is the vector field $\nabla_X v$ which expressed in coordinates $\{x^\alpha\}_{\alpha=0}^3$ (thus with respect to a coordinate basis $\{\partial_\alpha\}_{\alpha=0}^3$ is given by

$$(\nabla_X v)^\alpha = X^\mu (\nabla_\mu v)^\alpha$$

where $(\nabla_\mu v)^\alpha$ is the α-component of the covariant derivative of v in the direction of $\frac{\partial}{\partial x^\mu}$ (i.e., we abbreviate $\nabla_\mu = \nabla_{\frac{\partial}{\partial x^\mu}}$) defined by

$$(\nabla_X v)^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\lambda} v^\alpha v^\lambda$$

where $\Gamma^\alpha_{\mu\lambda}$ are the Christoffel symbols of g, defined by

$$\Gamma^\alpha_{\mu\lambda} = \frac{1}{2} g^{\alpha\tau} (\partial_\mu g_{\lambda\tau} + \partial_\lambda g_{\mu\tau} - \partial_\tau g_{\mu\lambda})$$

If f is a scalar function, we also define

$$\nabla_X f = X^\mu \partial_\mu f$$

(so the covariant derivative of a scalar agrees with the "calculus directional derivative". In particular, $\nabla_\mu f = \partial_\mu f$)

Remark 3.11. It is an exercise in tensor calculus to show that $\nabla_X v$, as introduced above, is well-defined, i.e., it is independent of the coordinate system we use.

3.3. **Crucial observation about notation.** Throughout the literature, one always writes,$\nabla_\mu v^\alpha$ for, $(\nabla_\mu v)^\alpha$) i.e., $\nabla_\mu v^\alpha = (\nabla_\mu v)^\alpha$. Thus, $\nabla_\mu v^\alpha$ is the α-component of the covariant derivative of v in the direction of $\frac{\partial}{\partial x^\mu}$, and not the covariant derivative of the α-component of v in the direction of $\frac{\partial}{\partial x^\mu}$.

The way we introduced covariant differentiation seems very ad hoc because of the pragmatic approach we are taking here. Students should consult the suggested literature for a more elegant and natural way of doing it.

The following proposition summarizes the basic properties of the covariant derivative. For convenience, some properties are stated in coordinates and in a coordinate-free fashion.

Proposition 3.12. *For vector fields X,Y, and Z, and scalar function f and h, it holds that:*

(a) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

(b) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

(c) (product rule)

$\nabla_X(g_{\alpha\beta}Y^\alpha Z^\beta) = g_{\alpha\beta}\nabla_X Y^\alpha Z^\beta + g_{\alpha\beta}Y^\alpha \nabla_X Z^\beta$,

$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Note that the product rule would not hold if we were taking an ordinary derivative instead of a covariant derivative, e.g., $\partial_\mu(g_{\alpha\beta}Y^\alpha Z^\beta) = g_{\alpha\beta}\partial_\mu Y^\alpha Z^\beta + g_{\alpha\beta}Y^\alpha\partial_\mu Z^\beta + \partial_\mu g_{\alpha\beta}Y^\alpha Z^\beta$

(d) (torsion-free condition)

$\nabla_\mu\nabla_v f - \nabla_v\nabla_\mu f = 0$

$\nabla_X Y - \nabla_Y X = [X,Y]$

where $[X,Y]$ is a vector field called the commutator of X and Y defined as follows: $X = X^\alpha\partial_\alpha$, $Y = Y^\beta\partial_\beta$

$$[X,Y] = X^\alpha\partial_\alpha Y^\beta\partial_\beta - Y^\beta\partial_\beta X^\alpha\partial_\alpha = (X^\alpha\partial_\alpha Y^\beta - Y^\beta\partial_\alpha X^\beta)\partial_\beta = [X,Y]^\alpha$$

where it can be shown that $[X,Y]$ is independent of the coordinate system used.

Property (c) is also known as compatibility between the covariant derivative and the metric. While it is possible to define other derivative operators, one can show that there exists a unique derivative operator satisfying (a) through (d) above. It is called the Levi-Civita covariant derivative or Levi-Civita connection (covariant derivatives are also known as "connections").

3.4. **Duality and one-forms.**

Definition 3.13. A one-form in $\mathbb{R}^4$ is a linear map that assigns to each $x \in \mathbb{R}^4$ a linear map $\omega(x) : \mathbb{R}^4 \to \mathbb{R}$

If we define the maps $dx^\alpha : \mathbb{R}^4 \to \mathbb{R}$ by

$$dx^\alpha(\frac{\partial}{\partial x^\beta}) = \delta^\alpha_\beta$$

extending this definition linearly to all vectors, then a one-form ω can be expressed as

$$\omega = \omega_\alpha dx^\alpha$$

where the ω_α are functions that are the components of ω in these coordinates. Given a vector field v, we can define a one-form v_b (real "v-flat") by

$$\bar{v}_b(X) = g(v, X)$$

For any vector field X. It is not difficult to see that the components of v_b are given by

$$(v_b)_\alpha = g_{\alpha\beta} v^\beta.$$

v_b is called the one-form dual to v. Similarly, given a one-form ω, we define the vector field $\omega^\#$(real "ω-sharp") by

$$g\left(X, \omega^\#\right) = \omega(X)$$

for any vector field X (which is well defined in view of the non-degeneracy of g). It follows that in components

$$\left(\omega^\#\right)^\alpha = g^{\alpha\beta}\omega_\beta$$

$\omega^\#$ is called the vector field dual to ω.

The maps $(\cdot)_b$ and $(\cdot)^\#$ are inverse of each other and provide isomorphisms between space of vector fields and forms:

$$g(X, (v_b)^\#) = v_b(X) = g(X, v) \Rightarrow g(X, (V_b)^\# - v) = 0 \text{ for all } X$$

$$\text{and } (\omega^\#)_b(X) = g(\omega^\#, X) = \omega(X) \Rightarrow ((\omega^\#)_b - \omega)(X) = 0 \text{ for all } X$$

In view of the above, we can identify v and ω with their duals. Therefore, we will no longer write b and # (it will be clear from the context whether we are dealing with a vector field or a one-form). In components, an upper index indicates a vector field and a lower index a one-form. Thus

$$v_\alpha = g_{\alpha\beta} v^\beta \quad \textit{and} \quad \omega^\alpha = g^{\alpha\beta}\omega_\beta$$

Because of the above formulas, the operations of passing from a vector field to a form and vice-versa are known as raising and lowering indices (lowering an index: vector field $\mapsto$ form; raising an index: form $\mapsto$ vector field). We can also use these isomorphisms to define an inner product between one-forms ω and μ by

$$g(\omega, \mu) = g_{\alpha\beta}\omega^\alpha\mu^\beta = g^{\alpha\beta}\omega_\alpha\mu_\beta$$

Where the last equality follows from a simple calculation. Moreover, for one-forms or vector fields: $g(v, \omega) = g_{\alpha\beta} v^\alpha \omega^\beta = v^\alpha\omega_\alpha = v_\alpha\omega^\alpha = g^{\alpha\beta} v_\alpha \omega_\beta$.

We will now extend covariant differentiation to forms. We do this by demanding it to satisfy a product rule.

Definition 3.14. The covariant derivative of a one-form ω in the direction of a vector field X is defined as the one-form $\nabla_X\omega$ given by

$$\nabla_X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)$$

For any vector field Y. Using the definition of $\nabla_X Y^\alpha$ we find

$$\nabla_\mu\omega_\alpha = \partial_\mu\omega_\alpha - \Gamma^\lambda_{\mu\alpha}\omega_\lambda$$

Where, similarly to what we had for vector fields, $\nabla_\mu\omega_\alpha$ means $(\nabla_{\frac{\partial}{\partial x^\mu}} W)_\alpha$. The product rule holds for $\nabla_\mu(f\omega_\alpha)$, f a function.

3.5. **Tensors.** We define the linear map $dx^\alpha \otimes dx^\beta : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$, called the tensor product of $dx^\alpha \otimes dx^\beta$, by

$$dx^\alpha \otimes dx^\beta(X,Y) = dx^\alpha(X)dx^\beta(Y).$$

Using this expression we can generalize one-forms, forming maps that act on an on ordered pairs of vector fields. A two-tensor T is defined, relative to coordinates, by the map

$$T = T_{\alpha\beta} dx^\alpha \otimes dx^\beta,$$

where the $T_{\alpha\beta}$, called the components of T in these coordinates, are functions. T acts on $X = X^\alpha \partial_\alpha$ and $Y = Y^\alpha \partial_\alpha$

$$T(X,Y) = T_{\alpha\beta} dx^\alpha \otimes dx^\beta(X,Y) = T_{\alpha\beta} X^\alpha Y^\beta.$$

T is called symmetric if $T_{\alpha\beta} = T_{\beta\alpha}$

Arguing similarly to what we did for one-forms, we can extend covariant differentiation to two-tensors, leading to the following expression in coordinates

$$\nabla_\gamma T_{\alpha\beta} = \partial_\gamma T_{\alpha\beta} - \Gamma^\lambda_{\gamma\alpha} T_{\alpha\beta} - \Gamma^\lambda_{\gamma\beta} T_{\alpha\lambda},$$

where, as above, $\nabla_\gamma T_{\alpha\beta} = (\nabla_\gamma T)_{\alpha\beta}$.
It can be showed that these definitions do not depend on the system of coordinates one uses. We will also encounter two-tensors, that are tensor products of one-forms, i.e., $T_{\alpha\beta} = W_\alpha \mu_\beta$, in which case $\nabla_\gamma T_{\alpha\beta}$ can also be computed by the product rule:

$$\nabla_\gamma T_{\alpha\beta} = \nabla_\gamma (\omega_\alpha \mu_\beta) = \nabla_\mu \omega_\alpha \mu_\beta + \omega_\alpha \nabla_\gamma \mu_\beta.$$

From these definitions, we see that the matric is a symmetric two-tensor:

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta.$$

Compatibility of covariant differentiation with the metric becomes:

$$\nabla_\gamma g_{\alpha\beta} = 0.$$

Given a two-tensor, its trace is the function

$$\mathrm{tr}(T) = g^{\alpha\beta} T_{\alpha\beta}.$$

Again, the result does not depend on the system of coordinates. Note that

$$\partial_\mu \, \mathrm{tr}(T) = \nabla_\mu \mathrm{tr}(T) = g^{\alpha\beta} \nabla_\mu T_{\alpha\beta}.$$

The divergence of a vector field v is the function $\mathrm{div}(v)$ defined as

$$\mathrm{div}(v) = g^{\alpha\beta} \nabla_\alpha v_\beta = \nabla_\alpha v^\alpha.$$

We can also take the divergence of a two-tensor: it is the one-form div(T) defined as

$$\mathrm{div}(T)_\beta = \nabla_\alpha T^\alpha{}_\beta,$$

where $T^\alpha_\beta = g^{\alpha\beta} T_{\gamma\beta}$ is $T_{\alpha\beta}$ with the first index raised (see below).

The covariant wave operator $\square_g$ applied to a scalar function f is defined by any of the following equivalent expressions:

$$\begin{aligned}\Box_g f &= g^{\alpha\beta}\nabla_\alpha\nabla_\beta f\\ &= \nabla^\alpha\nabla_\alpha f, \text{ where by definition } \nabla^\alpha = g^{\alpha\beta}\nabla_\beta\\ &= \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}g^{\alpha\beta}\partial_\beta f\right)\\ &= g^{\alpha\beta}\partial_\alpha\partial_\beta f - g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma f.\end{aligned}$$

We can also define a covariant wave operator applied to vector fields and tensors by $\nabla^\alpha\nabla_\alpha V^\beta$, $\nabla^\alpha\nabla_\alpha T_{\beta\gamma}$, etc.

3.6. **Some further remarks on tensors etc.** As before, our pragmatic approach leads to somewhat ad hoc definition of two-tensors, their covariant derivative and their trace, but this will suffice to our purposes. The above concepts cover almost all the geometric background we will need. Here, we introduce a few more ideas that will occasionally be needed, and make some observations.

In the terminology, "two" refers to the fact that T acts on two vector fields, although we can let T act in one vector field, resulting in a one-form:

$$T(X,\cdot) = T_{\alpha\beta}dx^\alpha \otimes dx^\beta(X,\cdot) = X^\alpha T_{\alpha\beta}dx^\beta$$

We can also consider $T(\cdot, X)$, which in general will be different than $T(X,\cdot)$ unless T is symmetric.

Strictly speaking, our definition of two-tensors is that of a covariant two-tensor, covariant here referring to the fact that it acts on vectors. We can also have vector fields act as one-forms in the same way as one-forms act on vector fields, i.e. we define

$$\frac{\partial}{\partial x^\alpha}\left(dx^\beta\right) = \delta^\beta_\alpha$$

and extend this definition linearly to have $\frac{\partial}{\partial x^\alpha}$ act on any one-form. We can then define the tensor product of ∂_α and ∂_β by

$$\partial_\alpha \otimes \partial_\beta(\omega,\mu) = \partial_\alpha(\omega)\partial_\beta(\mu)$$

for any two one-forms ω and μ. (Note: $\partial_\alpha(\omega)$ is defined above, it is not the derivative of ω; instead, $\omega = \omega_\alpha dx^\alpha$, $\partial_\alpha(\omega) = \partial_\alpha(\omega_\beta dx^\beta) = \omega_\beta\partial_\alpha(dx^\beta) = \omega_\beta\delta^\beta_\alpha = \omega_\alpha$.) We can then define a contravariant two-tensor by

$$T = T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta,$$

which acts on pairs of one-forms. T is called symmetric if the functions $T^{\alpha\beta}$ are symmetric.

Defining the tensor product of vector fields and one-forms in the obvious way, we can form mixed contravariant covariant tensors. For example, a 1-contravariant 1-covaraint tensor is

$$T = T^\alpha{}_\beta\partial_\alpha \otimes dx^\beta,$$

which acts on a pair (ω, X) of a one-form and one vector field.

Obviously, there is no need to restrict ourselves to two-tensor, (i.e., tensors that act on pairs of objects). A k-contravariant and l-covariant tensor, or a (k,l) tensor for short (where (k,l) is called the rank of the tensor), is given by

$$T = T^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_k}\partial_{\alpha_1}\otimes\cdots\otimes\partial_{\alpha_k}\otimes dx^{\beta_1}\otimes\cdots\otimes dx^{\beta\ell}$$

which acts on $(\underbrace{\omega_1,\cdots,\omega_k,}_{k \text{ one-forms}}\underbrace{X_1,\cdots,X_\ell}_{l \text{ vector fiedds}})$

For our purposes, the whole distinction between covariant and contravariant tensors is immaterial, as we can use the isomorphism between one-forms and vector fields to pass from one to the other. For example, the $(0,2)$ tensor

$$T = T_{\alpha\beta} dx^\alpha \otimes dx^\beta$$

can be identified with the $(1,1)$ tensor

$$T = T^\alpha{}_\beta \partial_\alpha \otimes dx^\beta$$

or yet with the $(2,0)$ tensor

$$T = T^{\alpha\beta} \partial_\alpha \otimes \partial_\beta,$$

where $T^\alpha_\beta = g^{\alpha\gamma} T_{\gamma\beta}$ and $T^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} T_{\gamma\delta}$. Thus, a (k,l) tensor can be thought of as a $(k+1, \ell-1)$ tensor etc.

We note that for tensors that are not symmetric, we have to pay attention to the order of the indices when they are raised and lowered. E.g., if we write T^α_β, it is not clear if it means $g^{\alpha\gamma} T_{\gamma\beta}$ or $g_{\beta\gamma} T^{\gamma\alpha}$. Both expressions agree if T is symmetric since

$$g^{\alpha\gamma} T_{\gamma\beta} = g^{\alpha\gamma} g_{\gamma\delta} g_{\beta\tau} T^{\delta\tau} = \delta^\alpha_\delta g_{\beta\tau} T^{\delta\tau} = g_{\beta\tau} T^{\alpha\tau}$$

but are otherwise different. With the proper care with the order of the indices, we can always raise and lower indices, and do not need to keep the distinction between covariant and contravariant tensor.

Using these ideas we can also write the trace as

$$\mathrm{tr}(T) = g^{\alpha\beta} T_{\alpha\beta} = T^\alpha{}_\alpha,$$

which we can write simply $T^\alpha{}_\alpha$ if T is symmetric.

A sum over an upper and a lower index is called a contraction. E.g. in the expression $T^\alpha{}_{\beta\alpha\gamma}$ we are contracting the first index with the third. Because this can also be written as

$$T^\alpha{}_{\beta\alpha\gamma} = g^{\alpha\delta} T_{\alpha\beta\delta\gamma}.$$

Contractions are sometimes also called traces, although for an arbitrary (γ, ℓ) tensor we have to specify which indices are being traced (i.e., contracted).

Notation 3.15. We often make an abuse of language and reference to the components of a tensor, e.g., $T_{\alpha\beta}$, as "the tensor."

The above constructions also allow us to construct new tensors out of old ones. E.g, if $T = T_{\alpha\beta} dx^\alpha \otimes dx^\beta$ and $U = U^\alpha{}_\beta \partial_\alpha \otimes dx^\beta$, then $V = T \otimes U$ is given by

$$V = \left(T_{\alpha\beta} dx^\alpha \otimes dx^\beta \right) \otimes \left(U^\gamma{}_\delta \partial_\gamma \otimes dx^\delta \right) = \underbrace{T_{\alpha\beta} U^\gamma{}_\delta}_{= V_{\alpha\beta}{}^\gamma{}_\delta} dx^\alpha \otimes dx^\beta \otimes \partial_\gamma \otimes dx^\delta$$

and the product rule holds for such tensors:

$$\nabla_\mu V_{\alpha\beta}{}^\gamma{}_\delta = \nabla_\mu T_{\alpha\beta} U^\gamma{}_\delta + T_{\alpha\beta} \nabla_\mu U^\gamma{}_\delta$$

Remark 3.16. The geometric framework outlined so far is essential for those who want a solid understanding of relativistic fluids (after all, relativity is a geometric theory). However, students who do not have a background in geometry should still be able to follow the main ideas of these lectures. In this regard, if one thinks of a two-tensor T as a matrix whose entries are $T_{\alpha\beta}$, and of the covariant derivative ∇_μ as "the ordinary derivative ∂_μ + stuff that can usually be treated as lower order," then one will be able to follow much of what follows.

4. The relativistic Euler equation

The dynamics of a perfect (i.e., no viscous) relativistic fluid is described by the relativistic Euler equations to be introduced below.

Definition 4.1. The energy-momentum tensor of a relativistic perfect isotropic fluid in $\mathbb{R}^4$ is the symmetric two-tensor

$$T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta + p g_{\alpha\beta},$$

where $g_{\alpha\beta}$ is a Lorentzian metric, $\beta$and ϱ are real-valued functions representing the pressure and energy density of the fluid, u_α is a vector field (one-form, recall our identification) representing the (four-) velocity of fluid and normalized by

$$|u|_g^2 = g_{\alpha\beta}u^\alpha u_\beta = u^\alpha u_\alpha = -1$$

The energy-momentum tensor is a fundamental object that encodes the behavior of matter and is essential when one considers the interaction of gravity and matter (i.e., coupling to Einstein's equation). Each theory of matter (e.g., electromagnetism, elasticity, etc.) has its own energy momentum tensor (we will discuss more about this when we consider theories with viscosity). The fluid is called isotropic as we are assuming that if one is at rest with respect to the fluid then the stresses in all directions of the fluid are the same, although it is possible to construct fluid models without this assumption [20]. The fluid velocity is sometimes called the four-velocity to emphasize that in relativity the velocity is a vector field in spacetime (so it has four components). The assumption $|u|_g^2 = -1$ can be understood as follows. First, it says that u_α is timelike, so fluid particles do not travel faster than the speed of light. Second, the energy density ϱ entering in $T_{\alpha\beta}$ is the energy measured by an observer travelling with the fluid (i.e., at rest with respect to the fluid). It is possible to show, using kinetic theory, that the energy density measured by an observer with velocity v^α will be $v^\alpha v^\beta T_{\alpha\beta}$. Thus, for the fluid velocity itself we need to have $\varrho = u^\alpha u^\beta T_{\alpha\beta}$, thus $u^\alpha u_\alpha = -1$. Let us make another remark about kinetic theory: it also gives the above expression for $T_{\alpha\beta}$ as a"continuum limit" when viscosity is ignored (and under certain natural assumptions) [13]. While kinetic theory provides what is probably the best justification for defining $T_{\alpha\beta}$ by the above formula, it is also possible to postulate $T_{\alpha\beta}$ motivated by physical considerations [23].

Definition 4.2. The baryon density current of a relativistic perfect isotropic fluid is defined by

$$J^\alpha = n u^\alpha,$$

where n is a real valued function representing the baryon number density of the fluid and u^α is the fluid's velocity as above.

Physically, the baryon number density gives the density of matter of the fluid: the rest mass density (measured by an observer at rest w.r.t the fluid) is given by nm, where m is the mass of the baryonic particles that constitute the fluid (these are notions from kinetic theory [20]).

Notation 4.3. We will not deal with non-isotropic fluids so from now on we omit "isotropic".

Physically, the quantities p, ϱ, and n are not all independent and are related by a relation known as an equation of state (whose form is determined experimentally or from kinetic theory). Under "normal circumstances" (e.g., absent phase transitions), this relation is invertible in the sense that knowledge of any two quantities, e.g., ϱ and n, determines the third, e.g., p. In this case, we can choose any two of the three quantities to be the functional unknowns. We will choose here ϱ and n and assume that p is given as a function of these quantities, i.e.,$p = p(\varrho, n)$.

Definition 4.4. The relativistic Euler equations are defined by the equations:

$$\nabla_\alpha T^\alpha_\beta = 0, \quad \text{(conservslion of energy-momentum)}$$
$$\nabla_\alpha J^\alpha = 0, \quad \text{(conservslion of baryoic charge)}$$
$$p = p(\varrho, n), \quad \text{(equation of state)}$$

where $T_{\alpha\beta}$ and J^α are as above, $p = p(\varrho, n)$ is a given equation of state, ∇ is the covariant derivative of the metric $g_{\alpha\beta}$ figuring in $T_{\alpha\beta}$. Note that the fluid's velocity is normalized as in the definition of $T_{\alpha\beta}$.

Remark 4.5. On physical grounds we want $\varrho \geq 0$, $n \geq 0$ and, in most models, $p \geq 0$. From the point of view of the Cauchy problem, these should be assumed for the initial data and showed to propagate.

Remark 4.6. As said is the introduction, we can consider a relativistic fluid on a fixed background or coupled to Einstein's equations. In the first case, which will be treated in this section, we assume g given, but we keep track of derivatives of g for future application to Einstein's equation

We introduce the tensor $\Pi_{\alpha\beta}$ which corresponds to projection onto the space orthogonal to u. Explicitly:

$$\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta,$$

So that $\Pi_{\alpha\beta} u^\beta = u_\alpha + u_\alpha \underbrace{u_\beta u^\beta}_{=-1} = 0$, and if v is orthogonal to u we have $\pi_{\alpha\beta} v^\beta = v_\alpha + u_\alpha \underbrace{u_\beta v^\beta}_{=0} = v_\alpha$. We also note that $u^\alpha u_\alpha = -1$ implies

$$u^\alpha \nabla_\beta u_\alpha = 0.$$

It is convenient to decompose $\nabla_\alpha T^\alpha_\beta$ in the directions parallel and orthogonal to u. First:

$$\begin{aligned}\nabla_\alpha T^\alpha_\beta &= \nabla_\alpha((p+\varrho)u^\alpha u_\beta + p g_{\alpha\beta}) \\ &= u^\alpha \nabla_\alpha (p+\varrho) u_\beta + (p+\varrho)\nabla_\alpha u^\alpha u_\beta + (p+\varrho) u^\alpha \nabla_\alpha u_\beta + \nabla_\beta p,\end{aligned}$$

thus

$$\begin{aligned}u^\beta \nabla_\alpha T^\alpha_\beta &= -u^\alpha \nabla_\alpha (p+\varrho) - (p+\varrho)\nabla_\alpha u^\alpha + (p+\varrho) u^\alpha \underbrace{u^\beta \nabla_\alpha u_\beta}_{=0} + u^\beta V_\beta p \\ &= -u^\alpha \nabla_\alpha \varrho - (p+\varrho)\nabla_\alpha u^\alpha. \\ \Pi^{\gamma\beta}\nabla_\alpha T^\alpha_\beta &= u^\alpha \nabla_\alpha (p+\varrho) \underbrace{\Pi^{\gamma\beta} u_\beta}_{=0} + (p+\varrho)\nabla_\alpha u^\alpha \underbrace{\Pi^{\gamma\beta} u_\beta}_{=0} + (p+\varrho)\Pi^{\gamma\beta} u^\alpha \nabla_\alpha u_\beta + \Pi^{\gamma\beta}\nabla_\beta p \\ &= (p+\varrho) u^\alpha (\underbrace{g^{\gamma\beta}\nabla_\alpha u_\beta}_{=\nabla_\alpha u^\gamma} + u^\gamma \underbrace{u^\beta \nabla_\alpha u_\beta}_{=0}) + \Pi^{\gamma\beta}\nabla_\beta \varrho \\ &= (p+\varrho) u^\alpha \nabla_\alpha u^\gamma + \Pi^{\gamma\beta}\nabla_\beta p.\end{aligned}$$

Writing $\nabla_\alpha J^\alpha$ explicitly:$\nabla_\alpha J^\alpha = \nabla_\alpha (n u^\alpha) = u^\alpha \nabla_\alpha u + u \nabla_\alpha u^\alpha$.

Therefore we can rewrite the relativistic Euler equations as:

$$\begin{aligned}u^\alpha \nabla_\alpha \varrho + (p+\varrho)\nabla_\alpha u^\alpha &= 0, \\ (p+\varrho) u^\alpha \nabla_\alpha u^\beta + \Pi^{\beta\alpha}\nabla_\alpha p &= 0, \\ u^\alpha \nabla_\alpha u + u \nabla_\alpha u^\alpha &= 0.\end{aligned}$$

The first equation is the conservation of mass/energy, while the second equation is the conservation of momentum. These equations reduce to the non-relativistic Euler equations in the non-relativistic limit [20].

While it is not difficult to obtain local existence and uniqueness by writing the above equations as a first order symmetric hyperbolic system (see, e.g. [2] [5]), we will use a different approach due to Lichnerowicz [18] that makes the rile of the characteristics manifest and connects with what we will discuss later. In fact, as we will see, but also as expected physically, there are two types of propagation in the fluid: sound waves and transport of vorticity. These correspond to different characteristics and thus should be treated differently. The first order symmetric hyperbolic system, however, treats both at the same level.

Before continuing, we will need a few more notions.

4.1. **Thermodynamic properties of relativistic fluids.** We begin introducing the following quantities:

- The internal energy density E of the fluid:
$$\varrho = n(1+E)$$
(strictly speaking the factor n should be the rest mass density nm, see above, but there is no harm in setting $m = 1$ here). Thus, the energy density of the fluid takes into account the energy coming from the fluids rest mass.
- The specific enthalpy h of the fluid
$$h = \underbrace{p+\varrho}_{n}, \quad \text{assuming} \quad n > 0.$$
- We assume the existence of functions s and θ, called the entropy density and temperature of the fluid, such that the first law of thermodynamics holds:
$$dp = n\,dh – n\theta\,ds,$$
which can also be written
$$\begin{aligned} d\varrho &= hdn - un\theta ds, \\ dE &= -pd\left(\frac{1}{n}\right) + \theta ds. \end{aligned}$$

As before, we can choose which two functions among these thermodynamic quantities are independent. Later we will choose s and h (so p, u, θ, ϱ and E are functions of s and h.

With these definitions, we can write

$$\begin{aligned} T_{\alpha\beta} &= (p+\varrho)u_\alpha u_\beta + pg_{\alpha\beta} = nhu_\alpha u_\beta + pg_{\alpha\beta}, \quad \text{then} \\ \nabla_\alpha T^\alpha_\beta &= \nabla_\alpha(nhu^\alpha)u_\beta + nhu^\alpha\nabla_\alpha u_\beta + \nabla_\beta p, \quad \text{so} \\ u^\beta\nabla_\alpha T^\alpha_\beta &= -\nabla_\alpha(nhu^\alpha) + u^\beta\nabla_\beta p \\ &= -h\underbrace{\nabla_\alpha(nu^\alpha)}_{=0}\underbrace{-nu^\alpha\nabla_\alpha h + u^\beta\nabla_\beta p}_{=u^\alpha(\underbrace{-u\nabla_\alpha h + \nabla_\alpha p}_{=-u\theta\nabla_\alpha s})} \end{aligned}$$

Under the physically natural assumption $\theta > 0$, which we will hereafter assume, we conclude:
$$u^\alpha\nabla_\alpha s = 0.$$
Physical interpretation: the fluid motion is locally adiabatic, i.e., entropy is constant along the flow lines of the fluid.

4.2. **Relativistic vorticity.** A very important quantity in fluids is the vorticity. In classical physics, it is the curl of the velocity. Since the curl in 3d can be identified (using Hodge duality) with the exterior derivative of the velocity (thought of as a one-form), it seems natural to define the vorticity of a relativistic fluid (where we are in four dimensions) as the exterior derivative of the four-velocity u. This is "almost" right, but the "correct" definition requires an adjustment.

Definition 4.7. The enthalpy current w is defined as

$$w^\alpha = hu^\alpha.$$

The vorticity Ω is defined as the two-form dw. In components it is given by the equivalent expressions:

$$\begin{aligned}\Omega_{\alpha\beta} &= \partial_\alpha(hu_\beta) - \partial_\beta(hu_\alpha)\\ &= \nabla_\alpha(hu_\beta) - \nabla_\beta(hu_\alpha).\end{aligned}$$

One reason to define the vorticity as above (rather than, say, du) is to have a relativistic version of Kelvin's circulation theorem. For a classical fluid with velocity v, we define its circulation along a closed loop γ as

$$\mathcal{C}_{\text{cl.}} = \oint_\gamma v \cdot d\ell.$$

Kelvin's theorem states that this quantity is conserved along fluid lines, i.e.,

$$D_t\mathcal{C}_{\text{cl.}} = (\partial_t + v\cdot\nabla)\mathcal{C}_{\text{cl.}} = 0$$

This theorem has such a clear physical interpretation as "conservation of vortices," that we expect something similar to hold for relativistic fluids. Indeed it does but the quantity that is conserved now is

$$\mathcal{C} = \oint_\gamma w_\alpha dx^\alpha = \oint_\gamma hu_\alpha dx^\alpha.$$

with this definition;

$$u^\mu\nabla_\mu\mathcal{C} = 0$$

The same way that the classical proof, goes through using dv, which is the vorticity, the relativistic version involves $d(hu)$, leading to a natural definition of the vorticity as we did. So, [20] for details.

Next, we derive an important relation between the vorticity and the entropy. Direct computation gives

$$\begin{aligned}u^\alpha\Omega_{\alpha\beta} &= u^\alpha(h\nabla_\alpha u_\beta + \nabla_\alpha h u_\beta - h\nabla_\beta u_\alpha - \nabla_\beta h u_\alpha)\\ &= h \underbrace{u^\alpha\nabla_\alpha u_\beta}_{=-\frac{1}{p+\varrho}\Pi^\alpha_\beta\nabla_\alpha p = \frac{1}{nh}\Pi^\alpha_\beta\nabla_\alpha p \quad \text{by} \quad \Pi^{\gamma\beta}\nabla_\alpha T^\alpha_\beta = 0} + u_\beta u^\alpha\nabla_\alpha h + \nabla_\beta h\\ &= -\frac{1}{n}\Pi^\alpha_\beta\nabla_\alpha p + u_\beta u^\alpha\nabla_\alpha h + \nabla_\beta h\\ &= \underbrace{-\frac{1}{n}\nabla_\beta p + \nabla_\beta h}_{=\theta\nabla_\beta s} - u_\beta\underbrace{(\frac{1}{n}u^\alpha\nabla_\alpha p - u^\alpha\nabla_\alpha h)}_{=-u^\alpha\nabla_\alpha s = 0}\end{aligned}$$

Therefore:

$$u^\alpha\Omega_{\alpha\beta} = \theta\nabla_\beta s.$$

This Equation is known as the Lichnerowicz equation. It implies that for an irrotational fluid, i.e., a fluid with $\Omega = 0$, the entropy must be constant, a result with no analogue in classical physics.

4.3. **Local existence and uniqueness.** We will rewrite the relativistic Euler equations as a system for w, Ω, b, and s. We assume that p, n, θ, and E are known functions of h and s.

We begin with an evolution equation for the vorticity. We can write the Lichnerowicz equation as (after multiplying by h)

$$\iota_w \Omega = h\theta ds,$$

where ι_w is the interior contraction of the two-form Ω with w, given by

$$(\iota_w \Omega)_\alpha = w^\mu \Omega_{\mu\alpha}.$$

Taking the exterior derivative:

$$d(\iota_w \Omega) = d(h\theta) \wedge ds,$$

where we used that $d^2 = 0$, and $\wedge$ is the wedge product of forms, which for one-forms is simply

$$\begin{aligned} w \wedge \mu &= (w_\alpha dx^\alpha) \wedge (\mu_\beta dx^\beta) = w_\alpha \mu_\beta dx^\alpha \wedge dx^\beta \\ &= \sum_{\alpha<\beta} (w_\alpha \mu_\beta - \mu_\beta w_\alpha) dx^\alpha \wedge dx^\beta \\ &\text{since} \quad dx^\alpha \wedge dx^\beta = -dx^\beta \wedge dx^\alpha. \end{aligned}$$

We now recall the following formula for the Lie derivative of a form in the direction of a vector field X:

$$\mathscr{L}_X \mu = d(i_X \mu) + i_X(d\mu).$$

In our case, $d\Omega = 0$ since $\Omega = dw$, so

$$\mathscr{L}_w \Omega = d(h\theta) \wedge ds.$$

Using the formula for the Lie derivative in terms of covariant derivatives, expanding the RHS and writing everything in components gives:

$$\begin{aligned} &w^\mu \nabla_\mu \Omega_{\alpha\beta} + \nabla_\alpha w^\mu \Omega_{\mu\beta} + \nabla_\beta w^\mu \Omega_{\alpha\mu} \\ &= \nabla_\alpha (h\theta) \nabla_\beta s - \nabla_\beta (h\theta) \nabla_\alpha s, \end{aligned}$$

which is our evolution equation for the vorticity.

This equation is remarkable because of the following. From the momentum equation we have $u^\alpha \nabla_\alpha u \sim \partial p \sim \partial s$. Commuting with it to get w we have $u^\alpha \nabla_\alpha w \sim \partial s, \partial h$. Since $\Omega \sim \partial_w$, we would thus naively expect $u^\alpha \nabla_\alpha \Omega \sim \partial^2 s, \partial^2 h$.However, this does not happen: the structure of the Lichnerowicz equation (which in particular casts ∂s as an exact derivative ds) leads to only one derivative on the RHS. This "gain of derivative" will help with existence and uniqueness below.

In particular, we point out how the first law of thermodynamic was used is the derivation of the vorticity equation; we did not simply apply $u^\mu \nabla_\mu$ to Ω and used $\nabla_\alpha T^\alpha_\beta = 0$.

Before continuing, let us consider an application. As seen, a necessary condition for irrotationality is that s is a constant. In fact, we have:

Proposition 4.8. *If s is constant and $\Omega = 0$ on $\{t = 0\}$, then s is constant and $\Omega = 0$ for $t > 0$.*

Proof. Integrating $u^\alpha \nabla_\alpha s = 0$ along the flow lines of s gives that $s =$ constant on spacetime. Thus, the equation for the vorticity gives

$$\mathscr{L}_w \Omega = 0,$$

which is a homogeneous transport equation for Ω. Since $\Omega \,|_{t=0} = 0$, uniqueness gives $\Omega = 0$. □

Remark 4.9. Of course, when we say $\Omega = 0$ for $t > 0$, we are referring to t belonging to an interval where the solution exists.

Next we derive an evolution equation for w. We start with the Hodge-Laplacion (not really a Laplacion because g is Lorentzian) of w:

$$\Box_H w = (dd^* + d^* d) w = dd^* w + d^* \Omega,$$

where d^* is the adjoint of d. Since $d^* w = -\nabla_\alpha w^\alpha$, compute:

$$\begin{aligned} d^* w = & - \nabla_\alpha w^\alpha = -\nabla_\alpha (h u^\alpha) = -u^\alpha \nabla_\alpha h - h \underbrace{\nabla_\alpha u^\alpha}_{= -\frac{u^\alpha \nabla_\alpha u}{u}} \\ = & - u^\alpha \nabla_\alpha h + \frac{h}{u} u^\alpha \nabla_\alpha n \\ = & - w^\alpha (\frac{\nabla_\alpha h}{h} - \frac{u^\alpha \nabla_\alpha u}{u}) = \iota_w dF, \end{aligned}$$

where $F = log \frac{n}{h}$. Thus

$$dd^* w = d(\iota_w dF) = \mathscr{L}_w dF.$$

It will be convenient to introduce $\widetilde{h} = h^2$ and consider $F = F(\widetilde{h}, s)$. Then, since $w^\alpha w_\alpha = -h^2$

$$\begin{aligned} \nabla_\alpha F = & \frac{\partial F}{\partial \widetilde{h}} \nabla_\alpha \widetilde{h} + \frac{\partial F}{\partial s} \nabla_\alpha s = - \frac{\partial F}{\partial \widetilde{h}} \nabla_\alpha (w^\beta w_\beta) + \frac{\partial F}{\partial s} \nabla_\alpha s \\ = & -2 \frac{\partial F}{\partial \widetilde{h}} w^\beta \nabla_\alpha w_\beta + \frac{\partial F}{\partial s} \nabla_\alpha s \\ = & -2 \frac{\partial F}{\partial \widetilde{h}} w^\beta (\Omega_{\alpha\beta} + \nabla_\beta w_\alpha) + \frac{\partial F}{\partial s} \nabla_\alpha s \\ = & -2 \frac{\partial F}{\partial \widetilde{h}} w^\beta \nabla_\beta w_\alpha + 2 \frac{\partial F}{\partial \widetilde{h}} \underbrace{w^\beta \Omega_{\beta\alpha}}_{= h\theta \nabla_\alpha s} + \frac{\partial F}{\partial s} \nabla_\alpha s \\ = & -2 \frac{\partial F}{\partial \widetilde{h}} w^\beta \nabla_\beta w_\alpha + (2 \frac{\partial F}{\partial \widetilde{h}} h\theta + \frac{\partial F}{\partial s}) \nabla_\alpha s \end{aligned}$$

To simply the notation, we henceforth adopt:

Notation 4.10. We will use B to indicate a generic expressions (which can vary from line to line) depending on at most the number of derivatives of its arguments.

Using the formula for the Lie derivative in terms of covariant derivatives:

$$(\mathscr{L}_w dF)_\gamma = -\alpha \frac{\partial F}{\partial \widetilde{h}} w^\alpha w^\beta \nabla_\alpha \nabla_\beta C_\gamma + (2 \frac{\partial F}{\partial \widetilde{h}} h\theta + \frac{\partial F}{\partial S}) w^\alpha \nabla_\alpha \nabla_\gamma s + B_\gamma (\partial_g, \partial_s, \partial_w).$$

but

$$\begin{aligned} w^\alpha \nabla_\alpha \nabla_\gamma s = w^\alpha \nabla_\gamma \nabla_\alpha s = \nabla_\gamma (\overbrace{w^\alpha \nabla_\alpha s}^{=0}) - \nabla_\gamma w^\alpha \nabla_\alpha s \\ = B_\gamma (\partial_g, \partial_s, \partial_w), \end{aligned}$$

so

$$(\mathscr{L}_w dF)_\gamma = -2 \frac{\partial F}{\partial \widetilde{h}} w^\alpha w^\beta \nabla_\alpha \nabla_\beta C_\gamma + B_\gamma (\partial_g, \partial_s, \partial_w).$$

On the other hand

$$(\Box_H w)_\gamma = -g^{\alpha\beta} \nabla_\alpha \nabla_\beta w_\gamma + R_{\gamma\alpha} w^\alpha, \text{so}$$

$$-g^{\alpha\beta}\nabla_\alpha\nabla_\beta w_\gamma + R_{\gamma\alpha}w^\alpha = -2\frac{\partial F}{\partial \widetilde{h}} w^\alpha w^\beta \nabla_\alpha\nabla_\beta w_\gamma + (d^*\Omega)_\gamma + B_\gamma(\partial_g, \partial_s, \partial_w).$$

Compute:

$$\begin{aligned}
2\frac{\partial F}{\partial \widetilde{h}} &= 2\frac{\partial F}{\partial h}\underbrace{\frac{\partial h}{\partial \widetilde{h}}}_{=\frac{1}{2h}} = \frac{1}{h}\frac{\partial}{\partial h}\log\frac{n}{h} = \frac{1}{h}\left(\frac{1}{n}\frac{\partial n}{\partial h} - \frac{1}{n}\right) \\
&= -\frac{1}{h^2}\left(L - \frac{h}{n}\frac{\partial u}{\partial h}\right), thus \\
&\left(-g^{\alpha\beta} - \left(1 - \frac{h}{u}\frac{\partial u}{\partial h}\right)\frac{w^\alpha w^\beta}{h^2}\right)\nabla_\alpha\nabla_\beta w_\gamma \\
&= -R_{\gamma\alpha}w^\alpha + (d^*\Omega)_\gamma + B_\gamma(\partial_g, \partial_s, \partial_w).
\end{aligned}$$

Next, we apply $w^\mu\nabla_\mu$ to this equation and compute:

$$\begin{aligned}
w^\mu\nabla_\mu(d^*\Omega)_\gamma &= w^\mu\nabla_\mu\nabla_\nu\Omega^\nu{}_\gamma \\
&= \underbrace{w^\mu\nabla_\nu\nabla_\mu\Omega^\nu{}_\gamma}_{=\nabla_\nu(\underbrace{w^\mu\nabla_\mu\Omega^\nu{}_\gamma}_{=B_\gamma(\partial g,\partial w,\partial s,\partial h,\Omega)})-\nabla_v w^\mu\nabla_\mu\Omega^\nu{}_\gamma} + R_{\mu\nu}w^\mu\Omega^v{}_\gamma + R_{\gamma\nu\lambda}^{\ \ \mu}w^\lambda\Omega_\mu{}^\lambda \\
&= B_\gamma(\partial^3 g, \partial^2 w, \partial^2 s, \partial^2 h, \partial\Omega).
\end{aligned}$$

Thus

$$\begin{aligned}
&\left[g^{\alpha\beta} + \left(1 - \frac{h}{n}\frac{\partial u}{\partial h}\right)\frac{w^\alpha w^\beta}{h^\alpha}\right] w^\mu\nabla_\mu\nabla_\alpha\nabla_\beta w_\gamma \\
&\qquad = B_\gamma(\partial^2 g, \partial^2 w, \partial^2 s, \partial^2 h, \partial\Omega).
\end{aligned}$$

Let us suppose that $\frac{h}{n}\frac{\partial u}{\partial h} > 0$, so we can define $\frac{1}{z} = \frac{h}{n}\frac{\partial u}{\partial h}$ and the above becomes:

$$\begin{aligned}
&\left[zg^{\alpha\beta} - (1-z)\frac{w^\alpha w^\beta}{h^2}\right] w^\mu\nabla_\mu\nabla_\alpha\nabla_\beta w_\gamma \\
&\qquad = B_\gamma(\partial^3{}_g, \partial^2{}_w, \partial^2{}_s, \partial^2{}_h, \partial\Omega).
\end{aligned}$$

Proposition 4.11. *Let*

$$\begin{aligned}
G^{\alpha\beta} &= zg^{\alpha\beta} - (1-z)\frac{w^\alpha w^\beta}{h^2} \\
&= zg^{\alpha\beta} - (1-z)u^\alpha u^\beta
\end{aligned}$$

where $0 < z \le 1$ and $|u|_g^2 = -1$. Then, $G^{\alpha\beta}$ is an (inverse) Lorentzian matric, and the operator

$$G^{\alpha\beta}w^\gamma\partial_\alpha\partial_\beta\partial_\gamma = \left[zg^{\alpha\beta} - (1-E)\frac{w^\alpha w^\beta}{h^2}\right] w^\gamma\partial_\alpha\partial_\beta\partial_\gamma$$

is a strictly hyperbolic third order operator.

Proof. This can be verified, for example, by computing the characteristics associated to $G^{\alpha\beta}$ and $G^{\alpha\beta}w^\gamma$. □

We now consider the equations derived for s, Ω and w. In these equations, we treat h as a function of w by $h = \sqrt{-w^\alpha w_\alpha}$, and expand the covariant derivatives, absorbing the terms in the

Christoffel symbols into the terms on the RHS of the equations. Doing so, we find (we multiplied the equation for s by h):

$$\begin{aligned}
&w^\alpha \partial_\alpha s = 0, \\
&w^\mu \partial_\mu \Omega_{\alpha\beta} = B_{\alpha\beta}(\partial g, \partial w, \partial s, \Omega), \\
&\left[z g^{\alpha\beta} - (1-z)\frac{w^\alpha w^\beta}{\sqrt{-w^\mu w_\mu}} \right] w^\gamma \partial_\alpha \partial_\beta \partial_\gamma w_\delta = B_\delta(\partial^3 g, \partial^2 w, \partial^2 s, \partial\Omega)
\end{aligned}$$

And we assume that $0 < z \leq 1$ (we will justify this assumption later on). We use the notation $G^{\alpha\beta}$ for the term in bracket as in the above proposition and note that the order of derivatives appearing on the RHS is compatible with the order of this mixed system so that its characteristics are given simply by the characteristics of the operators on the LHS (recall that at this point g is considered given).

Thus the system's characteristics are determined by

$$w^\alpha \xi_\alpha = 0$$

which are the flow lines of w (or of u), and

$$G^{\alpha\beta} \xi_\alpha \xi_\beta = 0$$

which are the characteristics cones, (i.e., the analog of the light cone if G were the Minkowshi metric) of the metric G.

Denote by $\|\cdot\|_\mu$ the H^N-Sobolev norm in Ω^3.

Involving standard energy estimates for strictly hyperbolic operators (see, e.g.,[16],[17]), we obtain

$$\|s\|_N \lesssim \|s(o)\|_N + \int_0^t B(\|w\|_N, \|s\|_N),$$

$$\|\Omega\|_N \lesssim \|\Omega(o)\|_N + \int_0^t B(\|g\|_{N+1}, \|w\|_{N+1}, \|s\|_{N+1}, \|\Omega\|_N),$$

$$\|w\|_{N+2} \lesssim \|w(o)\|_{N+2} + \int_0^t B(\|g\|_{N+3}, \|w\|_{N+2}, \|s\|_{N+2}, \|\Omega\|_{N+1}),$$

where we use the following above of notations: when we estimate a term like $\|\partial^2 s\|_N$, the derivatives could be time derivatives, so we have $\|\partial^2 s\|_N \lesssim \|s\|_{N+2} + \|\partial_t s\|_{N+1} + \|\partial_t^2 s\|_N$. But from the point of view of derivative counting all terms contribute the same. Also, on the LHS, we should have $\|w\|_{N+2} + \|\partial_t w\|_{N+1} + \|\partial_t^2 w\|_N$, but all terms contribute as $\|w\|_{N+2}$. Switching N to $N+1$ in the estimate for s and $N+2$ to $N+1$ in the estimate for w, and defining

$$\mathscr{N} = \|s\|_{N+1} + \|\Omega\|_N + \|w\|_{N+1}$$

we obtain:

$$\mathscr{N} \lesssim \mathscr{N}(o) + \int_0^t \mathscr{N},$$

which implies the energy bound for small t:

$$\mathscr{N} \lesssim C(\mathscr{N}(o)).$$

This estimate is the main ingredient for a proof of local existence and uniqueness, similarly to the standard argument for non-linear wave equations (see the rough notes on non-linear wave equations).

Other elements for the proof are:

Under the above assumptions ($0 < z \leq 1, n, \theta > 0,$ etc.), it is possibly to successively solve for the time derivatives $\partial_t^k u, \partial_t^k s, \partial_t^k h$ in terms of the data. This implies (a) that we can construct initial data for the s, Ω, w system out of data for the original system, and (b) that we can construct analytic solutions to the original equations of motions. These analytic solutions satisfy the system for w, Ω, w with $\Omega_{\alpha\beta} = \partial_\alpha(hu_\beta) - \partial_\beta(hw_\alpha)$ and $w_\alpha = hu_\alpha$. Given non-analytic data to the original system, we approximate it by analytic data and use the energy bound (that holds to the analytic solutions) to obtain, via a limit, a non-analytic solution to the original equations of motion. In particular, we have a solution to

$$(p+\varrho)u^\alpha \nabla_\alpha u_\beta + \Pi^\alpha_\beta \nabla_\alpha p = 0,$$

where Π is, as before, the projection onto the orthogonal space to u, but we do not know yet it to have the form $\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ because we have not yet showed that $|u|_g^2 = -1$. However, for $(p+\varrho) > 0$(which will hold for small time, but see below for more), contracting with u^β :

$$u^\beta u^\alpha \nabla_\alpha u^\beta = \frac{1}{2} u^\alpha \nabla_\alpha (|u|_g^2) = 0,$$

thus u remains normalized if normalized initially.

Finally, uniqueness can also be proved with an energy estimate (in a lower norm) for the difference of the two solutions.

Let us now discuss the assumption $0 < z \leq 1$. Given an equation of state $p = p(\varrho, s)$, the fluid's sound speed defined as

$$c_s^\alpha = \left(\frac{\partial p}{\partial \varrho}\right)_s,$$

which is a well-defined quantity for physical equations of state since the pressure of a fluid cannot decrease with an increase in density. The sound speed is also given by the following equivalent expressions [20] :

$$\begin{aligned} c_s^2 &= \frac{1}{h}\left(\frac{dp}{dn}\right)_n = \frac{n}{h}\left(\frac{dh}{dn}\right)_s = \frac{1}{h}\left(\left(\frac{\partial p}{\partial n}\right)_E + \frac{dE}{dn}\left(\frac{\partial p}{\partial E}\right)_n\right) \\ &= \frac{1}{h}\left(\left(\frac{\partial p}{\partial n}\right)_E + \frac{p}{n^2}\left(\frac{\partial p}{\partial E}\right)_n\right). \end{aligned}$$

It follows that $z = c_s^2$. Thus, $0 < z \leq 1$ means that the fluid's sound speed is positive and no greater than the speed of light.

We conclude that the characteristic cones determined by $G^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ correspond to propagation of sound in the fluid. Thus, the characteristics of the relativistic Euler equations correspond to two types of propagation phenomena transport along the flow lines of u and sound waves (we identify $G^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ as waves because G is a Lorentzian metric).

We remark that N in the above estimates has to satisfy $N > \frac{2}{3}$, since we need to use Sobolev estimates and product estimates. From $u^\alpha \nabla_\alpha s = 0$, we obtain that s will remain positive if initially positive, and from $\nabla_\alpha J^\alpha = 0$, written as $u^\mu \nabla_\mu \log u = -\nabla_\mu u^\mu$, the same holds for u (provided, say, that the fluid's velocity does not blow up). Depending on the equation of state, from the theorem dynamic relations we obtain positivity of θ p and E. Putting it all together, we conclude:

Theorem 4.12. *Consider initial data in H^{N+1},$N > \frac{2}{3}$, for the relativistic Euler equations with an equation of state such that $s, h, \theta, n, E, p\,|_{t=0} > 0$, and such that $0 < c_s\,|_{t=0} \leq 1$. Assume also that $|u|_g^2 = -1$ at $t = 0$. Thus, there exists a unique classical solution to the relativistic Euler equations defined for time interval.*

Remark 4.13. We have written the relativistic Euler equations in a way that made its characteristics explicit and allowed us to prove existence and uniqueness. But the way we wrote them is not yet good for further applications, and we will present another form of writing the equations later on.

5. The Einstein-Euler system

We will now consider the relativistic Euler equations coupled to Einstein's equations

5.1. **Curvature.**

We begin with some definitions needed to define Einstein's equations.

Definition 5.1. The Riemann curvature tensor of a metric g is the four-tensor (a (1,3) tensor) given in a system of coordinates by

$$R_{\alpha\beta}{}^{\gamma}{}_{\delta} = \partial_\alpha \Gamma^\gamma_{\beta\delta} - \partial_\beta \Gamma^\gamma_{\alpha\delta} + \Gamma^\gamma_{\alpha\mu}\Gamma^\mu_{\beta\delta} - \Gamma^\gamma_{\beta\mu}\Gamma^\mu_{\alpha\delta}$$

where the Γ's are the Christoffel symbols.

The Ricci curvature tensor is the following two tensor given as a trace of the Riemann tensor.

$$R_{\alpha\beta} = g^{\mu v} R_{\mu\alpha v\beta} = R_{\mu\alpha}{}^{\mu}{}_{\beta}.$$

The scalar curvature is the trace of the Ricci tensor:

$$R = g^{\alpha\beta} R_{\alpha\beta} = R^\alpha{}_\alpha.$$

These expressions are well-defined in that they do not depend on the coordinates used.

Once again, those definitions seem ad hoc, and it is not clear what the above expressions have to do with what we intuitively expect as "curvature." This last concern is at least partially clarified by the following proposition:

Theorem 5.2. *(i)If $R_{\alpha\beta}{}^{\gamma}{}_{\delta}$ vanishes on an open set U, then on U the metric g is isometric to the Minkowski metric, i.e., g is the Minkowski metric on U, but not necessarily written w.r.t. standard rectangular coordinates.*

(ii)$R_{\alpha\beta}{}^{\gamma}{}_{\delta}$ measures the failure of the covariant derivatives to commute in the sense that

$$\nabla_\alpha \nabla_\beta X^\gamma - \nabla_\beta \nabla_\alpha X^\gamma = R_{\alpha\beta}{}^{\gamma}{}_{\delta} X^\delta,$$

for any vector field X

Remark 5.3. In Riemannian geometry, (i) holds with Minkowski replaced by Euclidean.

Since intuitively the Miskowski space is the canonical flat (i.e., non-curved) space, (i) shows a connection between our intuition of curvature and the Riemann tensor. As for (ii), we can imagine that measuring the rate of change of a quality along different "paths" (first the x^α direction and then in the x^β direction, and vice-versa) can lead to different results if such paths travel regions of space that are differently curved.

Proposition 5.4. *(i) The Ricci tensor is symmetric, i.e., $R_{\alpha\beta} = R_{\beta\alpha}$.*

(ii) The following identity holds:

$$\nabla_\alpha \left(R^\alpha_\beta - \frac{1}{2} R g^\alpha_\beta \right) = 0.$$

Proof. (i) follows by exploring certain symmetries of ${R_{\alpha\beta}}^{\gamma}{}_{\delta}$ that follow from its definition, e.g. ${R_{\alpha\beta}}^{\gamma}{}_{\delta} = -{R_{\beta\alpha}}^{\gamma}{}_{\delta}$ (which in particular imply that not all components of the Riemann tensor are independent; in fact, there are 20 independent components). (ii) follows from some further symmetries for covariant derivatives of the Riemann tensor (known as Bianchi identities). □

5.2. **Einstein's equation.**

Definition 5.5. Given an energy-momentum tensor $T_{\alpha\beta}$, Einstein's equations are defined as

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta},$$

where Λ is a constant known as cosmological constant. If $T_{\alpha\beta} = 0$, then we have the vacuum Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0.$$

Remark 5.6. Students should not be misled by the word "vacuum", which may suggest that spacetime is somewhat trivial, that "nothing is happening." This is definitely not the case, as one could imagine from how complicated $R_{\alpha\beta}$ and R are. Solutions to vacuum Einstein's equations can be quite complex and even develop singularities. In fact, some of the best known explicit solutions to Einstein's equations, like Schwarzschild and the Kerr solution, are solutions to vacuum Einstein's equations (with $\Lambda = 0$). Moreover, Christodovler proved [7] that singularities can form on solutions to vacuum Einstein's equations by the focusing of gravitational waves. (This does make sense physically: gravitational waves carry energy, so focusing them into a small region of spacetime can create a black hole.)

Remark 5.7. Here we are interested in the case where $T_{\alpha\beta}$ is the energy-momentum tensor for a perfect fluid, but our initial discussion applies as well to other theories (i.e. other energy-momentum tensors), so we will keep it general for now.

Notation 5.8. The energy momentum tensor involves variables that depend on the particular theory we are studying. In the case of relativistic Euler, as seen, then variables are, beside g that already appears on the LHS of Einstein's equations, u, ϱ, and p. But if we take, say, $T_{\alpha\beta}$ to be the energy-momentum of electromagnetism, then the variables in $T_{\alpha\beta}$ will be the electric and magnetic field E_m and B_m. In order to keep the discussion general, we will denote symbolically, all the variables in $T_{\alpha\beta}$ besides the metric g by ψ, and sometimes write $T_{\alpha\beta}(\psi)$ to indicate this. These variables are called the matter fields. We remark that "matter" means anything that is not gravity (i.e., all variables in $T_{\alpha\beta}$ except the metric). Thus, for example, if we have the electric and magnetic fields we call them "matter fields" even though physically we think of the electro-magnetic field in terms of radiation rather than matter.

As a consequence of the Bianchi identities we have (see proposition above) we have

$$\nabla_\alpha(R^\alpha_\beta - \frac{1}{2}Rg^\alpha_\beta + \Lambda g^\alpha_\beta) = 0,$$

thus $\nabla_\alpha T^\alpha_\beta = 0$ is a necessary condition for the existence of a solution to Einstein's equations. In particular, the equation of motion for the matter fields are $\nabla_\alpha T^\alpha_\beta = 0$. (This also gives another motivation for why the relativistic Euler equations are $\nabla_\alpha T^\alpha_\beta = 0$.)

5.3. **The Cauchy problem for Einstein's equations.** We will now discuss the Cauchy problem, roughly stated as (see below for a precise statement): given g and ψ initially, can we find g, ψ solving Einstein's equations and taking the initial data?

Suppose we have solution to Einstein's equations and consider coordinates $\{x^\alpha\}_{\alpha=0}^3$. Assume that initial data was given along $\Sigma = \{x^0 = t = 0\}$, and let N be unit future-directed (i.e. pointing toward $t > 0$) unit normal to Σ. Then using the expression for $R_{\alpha\beta}$ and R in coordinates we find that

$$(R_{\alpha\beta} - \frac{1}{2}Rg^\alpha_\beta + \Lambda g^\alpha_\beta)N^\alpha$$

involves no term with two derivatives of g w.r.t ∂_t. (This can be seen more easily in coordinates such that $g_{00} = -1, g_{0\alpha} = 0, N = (1,0,0,0)$.) Since the initial data involves prescribing g and $\partial_t g$ on Σ (since Einstein's equation are second order on g), we see that the initial data is constrained, i.e., it cannot be prescribed arbitrarily but it has to satisfy

$$(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta})N^\alpha = T_{\alpha\beta}N^\alpha \quad \text{on} \quad \Sigma$$

The existence of such constraints can be understood geometrically. If we have a solution to Einstein's equations and Σ is a hypersurface embedded in spacetime where initial data is given, then the induced metric on Σ (which agrees by assumption with the metric given as data) cannot be arbitrary but has to satisfy certain relations known as the Gauss-Codazzi equations. These equations also involves the second fundamental form of embedded in the spacetime. (Roughly, the second fundamental form relates the intrinsic geometry of Σ with that of the spacetime where it embeds.)

We also note that since Σ is three-dimensional, the initial metric given on Σ should be a metric on a three dimensional space, i.e., g_{ij} with nine components rather than $g_{\alpha\beta}$ with 16 components. On the other hand, we do want to solve Einstein's equations for the full spacetime metric (i.e., $g_{\alpha\beta}$ with 16 components.)

In view of our signature convention -+++, the metric given initially on Σ is Riemannian.

These considerations lead to the following definitions.

Definition 5.9. An initial data set for Einstein's equation consists of a three-dimensional manifold Σ, a Riemannian metric $\mathring{g}$ on Σ, a symmetric two-tensor $\mathring{k}$ on Σ, initial data $\mathring{\psi}$ for the matter fields, such that the Einstein's constraint equations, given by

$$R_{\mathring{g}} - |\mathring{k}|^2_{\mathring{g}} - (\text{tr}_{\mathring{g}}\mathring{k})^2 = 2\mathring{\varrho}$$
$$\nabla_{\mathring{g}}\text{tr}_{\mathring{g}}\mathring{k} - \text{div}_{\mathring{g}}\mathring{k} = \mathring{j}$$

are satisfied on Σ. Above, $R_{\mathring{g}}, \nabla_{\mathring{g}}, \text{tr}_{\mathring{g}}, \text{div}_{\mathring{g}}$ and $|\cdot|_{\mathring{g}}$ are respectively the scalar curvature, covariant derivative, trace, divergence, and norm of the metric $\mathring{g}$. The quantities $\mathring{\varrho}$ and $\mathring{j}$ are, respectively, a function an one-form on Σ with the property that $T(N,N) = \mathring{\varrho}$ and $T(N,\cdot) = \mathring{j}$(where T is the energy momentum tensor) whenever Σ embeds, with second fundamental form k, into a spacetime where Einstein's equations are satisfied.

The constraint equations are the relations needed to be satisfied by the initial data, as discussed above. The tensor $\mathring{k}$ plays the role of $\partial_t g\,|_{t=0}$: strictly speaking we cannot talk explicitly about $\partial_t g\,|_{t=0}$ since ∂_t is a coordinate dependent operator. Moreover, ∂_t would be transversal to Σ , but it does not make sense talk about transversality to Σ before having Σ embedded into a spacetime.

Definition 5.10. Solving Einstein's equation with a given initial set $I = (\Sigma, \mathring{g}, \mathring{h}, \mathring{\psi})$ consists of finding a four dimensional manifold M, a Lorentzian metric g, fields ψ, and an embedding $i : \Sigma \to M$, such that:

(i) Einstein's equations with $T_{\alpha\beta}(\psi)$ are satisfied in M.

(ii)$i^*(g) = \mathring{g}, i^*(\psi) = \mathring{\psi}$, where i^* is the pull-back via i.

(iii) the second fundamental form of the embedding $i : \Sigma \to M$ equals $\mathring{k}$.

Taking the trace of Einstein's equations and using that $g^{\alpha\beta} g_{\alpha\beta} = 4$ we find

$$-R + 4\Lambda = g^{\alpha\beta} T_{\alpha\beta}.$$

using this expression to substitute for R, we see that we can write Einstein's equations as

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} T_{\mu\nu} g_{\alpha\beta} + \Lambda g_{\alpha\beta},$$

which is more convenient for our purposes.

We will construct solutions to Einstein's equations for a given initial data set. We will consider $T_{\alpha\beta}$ for simplicity, as the ideas we will present apply to the case $T_{\alpha\beta} \neq 0$ as well. We henceforth assume an initial data set to be given

Embed Σ into $\mathbb{R} \times \Sigma$ and fix $p \in \Sigma$. We will initially construct a solution in a neighborhood of p. Consider coordinates $\{y^\alpha\}_{\alpha=0}^3$ defined as an open set about p, with $\{y^i\}_{i=1}^3$ coordinates on $\Sigma \cap U = \{y^0 = 0\}$. Assume that p corresponds to coordinates (0,0,0,0). In these coordinates, the Ricci tensor reads (using the formula for $R_{\alpha\beta}$):

$$R_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\mu \partial_\nu g_{\alpha\beta} + \partial_\alpha \partial_\beta g_{\mu\nu} - \partial_\alpha \partial_\mu g_{\beta\nu} - \partial_\beta \partial_\mu g_{\alpha\nu}) + F_{\alpha\beta}(g, \partial g),$$

where $F_{\alpha\beta}(g, \partial g)$ represents terms involving at most one derivative of g. We think of $R_{\alpha\beta}$ as a second order differential operator on g given by the above expression, and we want to solve $R_{\alpha\beta} = 0$. Thus we need to understand the operator $R_{\alpha\beta}$, so we look at its principal symbol: linearizing $R_{\alpha\beta}$ in the direction of a symmetric two tensor we find

$$\sigma(\text{Ricci})(h) = \frac{1}{2} g^{\mu\nu} (\xi_\mu \xi_\nu h_{\alpha\beta} + \xi_\alpha \xi_\beta h_{\mu\nu} - \xi_\alpha \xi_\beta h_{\beta\nu} - \xi_\beta \xi_\mu h_{\alpha\nu}).$$

Thus, if we take $h_{\alpha\beta} = \xi_\alpha \xi_\beta$ we find $\sigma(\text{Ricci})(h) = 0$ with $h \neq 0$. Therefore, every direction is characteristic for the Ricci operator, and we cannot solve $R_{\alpha\beta} = 0$ with $R_{\alpha\beta}$ given by the above expressions. (Note: exactly the same issue happens when one studies Ricci flow.) This degeneracy is of geometric character: it is a consequence of the fact that $R_{\alpha\beta}$ is invariant under diffeomorphisms: $\text{Ricci}(g) = \text{Ricci}(\varphi^*(g))$ for diffeomorphism φ . In other words, Einstein's equations turn out to an underdetermined system of PDEs.

We can remove the degeneracy of $R_{\alpha\beta}$ by choosing suitable coordinates, as follows. Define function $x^{(0)}, x^{(1)}, x^{(2)}$ and $x^{(3)}$ by solving the following initial value problem in U: (we write $^{(\alpha)}$ to indicate that the index in $x^{(\alpha)}$ is simply a label for these four function, it is not meant to be a tensor index), where we take to be functions of (y^0, y^1, y^2, y^3) :

$$\Box_g x^{(i)} = 0 \quad \text{in } U,$$
$$x^{(i)}(0, y^1 y^2 y^3) = y^i,$$
$$\frac{\partial}{\partial y^o} x^{(i)}(0, y^1 y^2 y^3) = 0.$$

for $i=1,2,3$ and

$$\begin{aligned} \square_g x^{(0)} &= 0 \quad \text{in } U,\\ x^{(0)}(0,y^1y^2y^3) &= 0,\\ \frac{\partial}{\partial y^o}x^{(0)}(0,y^1y^2y^3) &= 1, \end{aligned}$$

where we recall that $\square_g$ is the wave operator applied to a scalar function. Since the functions $x^{(\alpha)}$ agree with the coordinate functions y^α on $\Sigma\cap U$, we conclude that the $x^{(\alpha)}$ give rise to a system of coordinates on U(possibly shrinking U if needed), thus we write $x^{(\alpha)}=x^\alpha$, and now $\{x^\alpha\}_{\alpha=0}^3$ is a coordinate system about p. On the other hand, $\square_g x^{(\alpha)} = \square_g x^\alpha$ is coordinate independent, so we also have $\square_g x^\alpha = 0$ when $\square_g$ is expressed relative to the coordinates $\{x^\alpha\}_{\alpha=0}^3$, thus we have (using one of the expressions for $\square_g$):

$$\square_g x^\alpha = g^{\mu\nu}(x)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}x^\alpha - g^{\mu\nu}(x)\Gamma^\lambda_{\mu\nu}(x)\frac{\partial}{\partial x^\lambda}x^\alpha = 0,$$

where we write $g^{\mu\nu}(x)$ and $\Gamma^\lambda_{\mu\nu}(x)$ to emphasize that these are the metric and Christoffel symbol expressed relative to $\{x^\alpha\}_{\alpha=0}^3$ coordinates. But $\frac{\partial}{\partial x^\lambda}x^\alpha = \delta^\alpha_\lambda$ and $\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}x^\alpha = 0$, so we conclude

$$\Gamma^\alpha = 0$$

where $\Gamma^\alpha = g^{\mu\nu}\Gamma^\lambda_{\mu\nu}$.

Definition 5.11. The coordinates $\{x^\alpha\}_{\alpha=0}^3$, where $\Gamma^\alpha=0$, are called wave coordinates.

We stress that, by construction, wave coordinates depend on the metric g.

It can be showed that relative to wave coordinate,

$$g^{\mu\nu}(\partial_\alpha\partial_\beta g_{\mu\nu} - \partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\beta\partial_\mu g_{\alpha\nu}) = 0$$

so that $R_{\alpha\beta}=0$ reduces to

$$R_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + F_{\alpha\beta}(g,\partial g) = 0.$$

This is a system of quasi-linear equations which can be solved by standard techniques (see the rough notes on non-linear wave equations). However, the problem is that we are trying to prove existence of g, whereas wave coordinates required g to be given. To overcome this problem, we will do the following. We solve the equation that we know how to solve, i.e., $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta}+F_{\alpha\beta}(g,\partial g)=0$. Then, we try to show that this solution in fact solves Einstein's equations. Thus, it is convenient to introduce:

Definition 5.12. The reduced Ricci tensor of g is

$$R^H_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + F_{\alpha\beta}(g,\partial g),$$

defined in U. The reduced Einstein equations are

$$R^H_{\alpha\beta} = 0 \quad \text{in } U$$

We thus consider $R^H_{\alpha\beta}=0$ in U. Let us now provide initial conditions for this equation. Recall that the initial data is given on $\Sigma\cap U = \{y^0 = t = 0\}$.

Since we are given $\mathring{g}$ on Σ, we have $g_{ij}(0) = \mathring{g}_{ij}$. We need to prescribe $g_{0\alpha}(0)$, and we choose $g_{00}(0)=-1$, $g_{0i}(0)=0$. We also need to prescribe $\partial_t g_{\alpha\beta}(0)$. In suitable coordinates the second fundamental form is given by $k_{ij} = \frac{1}{2}\partial_t g_{ij}$, so we prescribe $\partial_t g_{ij}(0) = 2\mathring{k}_{ij}$. It remains to prescribe $\partial_t g_{0\alpha}(0)$. It can be verified that, combined with the choices we made so far, we can choose $\partial_t g_{0\alpha}$ such that our coordinates $\{y^\alpha\}_{\alpha=0}^3$ are wave coordinates on $\Sigma \cap U$, i.e., we can choose $\partial_t g_{0\alpha}$ so that

$$\Gamma^\alpha(0) = \Gamma^\alpha\,|_{\Sigma\cap 0} = 0.$$

Having prescribed initial data, we now obtain a solution $g_{\alpha\beta}$ to $R^H_{\alpha\beta} = 0$ in U, possibly after shrinking u. By continuity, $\Sigma \cap U$ is in fact a Lorentzian metric in U.

Now we want to show that g is a solution to Einstein's equation, i.e., its Ricci tensor satisfies $R_{\alpha\beta} = 0$. This will be the case if $\Gamma^\alpha = 0$ in U, since in this case the coordinates $\{y^\alpha\}_{\alpha=0}^3$ will in fact be wave coordinates for the metric g we found, in which case $R_{\alpha\beta} = R^H_{\alpha\beta}$ and then $R_{\alpha\beta} = 0$. We have that $\Gamma^\alpha = 0$ on $\Sigma \cap U$, so we have to prove that the vanishing of Γ^α on $\Sigma \cap U$ can be propagated to U.

Since g is a Lorentzian metric, its Riemann tensor satisfies the Bianchi identities, which imply, after some calculation:

$$\Box_g \Gamma^\alpha + H(\Gamma, \partial\Gamma) = 0 \quad \text{in } U$$

where $H(\Gamma, \partial\Gamma)$ represent terms involving at most one derivative of Γ. This is a homogeneous system of wave equations for Γ^α , for which one of the initial conditions is $\Gamma^\alpha(0) = 0$. Uniqueness for wave equations gives that

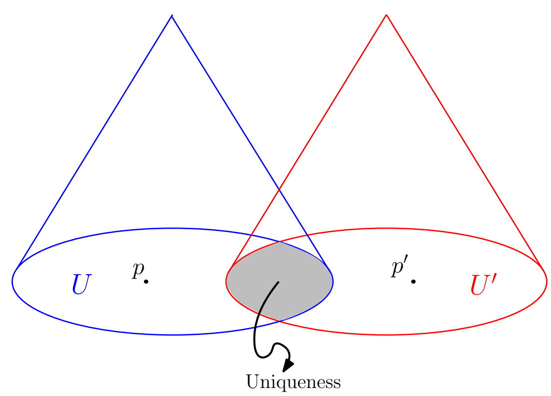
$$\Gamma^\alpha = 0 \quad \text{in } U \quad \text{if } \partial_t\Gamma^\alpha(0) = \partial_t\Gamma^\alpha\,|_{\Sigma\cap U} = 0.$$

We now invoke the following fact (see [5]):

Theorem 5.13. $\partial_t\Gamma^\alpha(0) = 0$ *if and only if the constraint equations are satisfied.*

Since in an initial data set the constraints are satisfied by assumption, we finally conclude that we have found a metric g in U such that $R_{\alpha\beta} = 0$ in U.

It remains to obtain a solution global in space, i.e., valid in a neighborhood of Σ in $\mathbb{R} \times \Sigma$. This can be done by using the domain of dependence property of the wave operator $g^{\mu\nu}\partial_\mu\partial_\nu$ and uniqueness for wave equations to give solutions constructed in different open sets U, U':



Caution: although the argument for gluing these solutions is not complicated, it is not as straightforward as the above picture may suggest, as we need to construct a system of coordinates valid on the intersection region in order to compare the two solutions from U and U'.

This beautiful result on existence of solutions to vacuum Einstein's equations was first proved by Choquet-Bruhat in 1952 [12], a result that can be considered the birth of mathematical general relativity (although Einstein himself did not seem to be impressed, see [Pa] p.291).

It is not difficult to see that plain uniqueness fails for $R_{\alpha\beta} = 0$. Let $M = (-\varepsilon, \varepsilon) \times \Sigma$ be a spacetime constructed as above, and $\varphi : M \to M$ be a diffeomorphism that is not the identity but agrees with the identity in a neighborhood of Σ. Then the metrics g and $\varphi^*(g)$ are two different metrics in M, but both inducing the same initial data on Σ, and both solving Einstein's equations since $\mathrm{Ricci}(g) = \mathrm{Ricci}(\varphi^*(g))$. (This does not contradict the uniqueness needed for the above gluing argument since there we are talking about uniqueness in wave coordinates, i.e., for the operator $g^{\mu\nu}\partial_\mu\partial_\nu$).

The problem with the above example is that the metrics g and $\varphi^*(g)$ are isometric. Thus, the manifolds (M, g) and $(M, \varphi^*(g))$ should not be distinguished in the category of Lorentzian manifolds. Thus, if we consider equivalence classes of manifolds, i.e., up to isometry, (M, g) and $(M, \varphi^*(g))$ are the same. Nevertheless, we can still produce non-uniqueness by consider a proper subset $M' \subset M$ that contains Σ, since (M', g) and (M, g) are not isometric.

It is possible to construct, however, the "largest" spacetime that solves Einstein's equations with the given initial data, called the maximal globally hyperbolic of the initial data, and this manifold is unique.

We can now investigate the Einstein Euler system. The above arguments depend essentially on properties of the Ricci tensor and geometric considerations, and it is not difficult to see that they apply to the case with matter as well, provided we can solve the coupled system in U. In the case of the Einstein-Euler system, we only need to add Einstein's equations to the system we have already derived for the relativistic Euler equations. We obtain:

$$\begin{aligned}
-\frac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu \delta_{\alpha\beta} &= B_{\alpha\beta}(\partial g, w, s) \\
w^\alpha \partial_\alpha s &= 0, \\
w^\mu \partial_\mu \Omega_{\alpha\beta} &= B_{\alpha\beta}(\partial g, \partial w, \partial s, \Omega), \\
\left[c_s^2 g^{\alpha\beta} - (1 - c_s^2) \frac{w^\alpha w^\beta}{\sqrt{-w^\alpha w_\alpha}} \right] w^\gamma \partial_\alpha \partial_\beta \partial_\gamma w_\delta &= B_\delta(\partial^3 g, \partial^2 w, \partial^2 s, \partial\Omega),
\end{aligned}$$

where we now write $z = c_s^2$. we can do energy estimates as before to find:

$$\|g\|_{N+\alpha} \lesssim \|g(0)\|_{N+\alpha} + \int_0^t B(\|g\|_{N+2}, \|w\|_{N+1}, \|s\|_{N+1}),$$

$$\|s\|_{N+1} \lesssim \|s(0)\|_{N+1} + \int_0^t B(\|w\|_{N+1}, \|s\|_{N+1}),$$

$$\|\Omega\|_N \lesssim \|\Omega(0\|_N + \int_0^t B(\|g\|_{N+1}, \|w\|_{N+1}, \|s\|_{N+1}, \|\Omega\|_N),$$

$$\|w\|_{N+1} \lesssim \|w(0)\|_{N+1} + \int_0^t B(\|g\|_{N+2}, \|w\|_{N+1}, \|s\|_{N+1}, \|\Omega\|_N),$$

and once again we observe that these estimates close, leading to existence of solutions. We leave the formulation of a precise statement of existence (and uniqueness in the above sense) to students (or see, e.g., [10]).

Remark 5.14. Because of the undetermined of Einstein's equations, we had a great deal of freedom in choosing our coordinates and initial data for the spacetime metric. This freedom, which was crucial to obtain solution, is known in the physics literature as gauge freedom.

Remark 5.15. The characteristics of the Einstein-Euler system are the same as the relativistic Euler equations plus the characteristics coming from Einstein's equations, namely, $g^{\alpha\beta}\xi_\alpha\xi_\beta$, which correspond to gravitational waves.

5.4. **Initial data.** We have not so far address the question of whether initial data sets do exist, i.e., whether it is possible to find initial data satisfying the constraints. This is a research problem in itself: we need to solve the constraint equations for $\mathring{g}$ and $\mathring{k}$. When appropriate formulated, the constraint equations turn out to be an elliptic system for $\mathring{g}$ and $\mathring{k}$, and the topology of Σ plays a role on whether or not this elliptic system admits solutions which in many situations of interest it does, this is not always the case. See [8].

6. New formulation of the relativistic Euler equations

The equations we derived in order to obtain local existence and uniqueness for the relativistic Euler equations involve operators that make the role of the characteristics manifest. Nevertheless, such equations are not yet good enough for more refined applications, such as the study of shock formation in the relativistic Euler equations. Here, we will present yet another way of writing the relativistic Euler equations. As we will explain, this new formulation of the equations exhibit several remarkable features, making it ameneable to certain applications in a way that other formulations are not.

6.1. **Auxiliary quantities.** We continue to use the same notation as before for the relativistic Euler equations, and here we introduce several new quantities that will be useful in what follows. Throughout, we denote by $\varepsilon^{\alpha\beta\gamma\delta}$ the totally antysymetric symbol nonanalized by $\varepsilon^{0123} = 1$.

Assumption 6.1. *For simplicity, in our new formulation of the relativistic Euler equations we will assume that the spacetime metric is the Minskowshi metric. The coordinates* $\{x^\alpha\}_{\alpha=0}$ *will be standard rectangular coordinates.*

Recall that c_s is the fluid's sound speed.

Definition 6.2. We introduce:

The (dimensionless) log-enthalpy:

$$\widehat{h} = \log(h/\overline{h}),$$

where $\overline{h}$ is some fixed reference constant value.

The u-orthogonal vorticity of a one-form :

$$\mathrm{vort}^\alpha(V) = -\varepsilon^{\alpha\beta\gamma\delta}u_\beta\partial_\gamma V_\delta.$$

The u-orthogonal vorticity vectorfield

$$\varphi^\alpha = \mathrm{vort}^\alpha(hu).$$

The entropy gradient one-form:

$$S_\alpha = \partial_\alpha s.$$

The modified vorticity of the vorticity:

$$C^\alpha = \mathrm{vort}^\alpha(\varphi) + c_s^{-2}\varepsilon^{\alpha\beta\gamma\delta}u_\beta\partial_\beta\widehat{h}\varphi_\delta + (\theta - \frac{\partial\theta}{\partial\widehat{h}})S^\alpha\partial_\lambda u^\lambda + (\theta - \frac{\partial\theta}{\partial\widehat{h}})u^\alpha S^\lambda\partial_\lambda\widehat{h} + (\theta - \frac{\partial\theta}{\partial\widehat{h}})S^\lambda g^{\alpha\beta}\partial_\lambda u_\beta$$

The modified divergence of the entropy gradient:

$$D = \frac{1}{n}\partial_\lambda S^\lambda + \frac{1}{n} S^\lambda \partial_\lambda \widehat{h} - \frac{1}{n} c_s^{-2} S^\lambda \partial_\lambda \widehat{h}.$$

The modified quantities C^α and D come about because of the following. In the applications we will discuss, we need to estimate vort(φ) and $\partial_\lambda S^\lambda$, but a good estimate is not available for these quantities. However, adding the right combination of variables to root and $\partial_\lambda S^\lambda$, we obtain quantities ($C^\alpha$ and D) that satisfy equations with a good structure for which estimates can be derived.

The n-orthogonal vorticity φ is related to Ω by duality: $\varphi^\alpha = 2u^\beta ({}^*\Omega)_{\mu\nu}$, where ${}^*\Omega$ is the Hode dual of Ω, given by $({}^*\Omega)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\Omega^{\mu\nu}$. The role of φ is to provide the vorticity "as a vector" rather than as a two form, as in the classified case.

Assumption 6.3. *In the previous definition, as well as in the ensuing discussion of the new formulation of the relativistic Euler equations, it is assumed that $\widehat{h}$ and s are the fundamental thermodynamic variables, with $h, n, \theta, \varrho, E,$ and p being functions of $\widehat{h}$ and s. We also assume our constructions to be such that $0 < c_s = c_s(\widehat{h}, s) \leq 1$.*

Definition 6.4. For $0 < c_s \leq 1$, the acoustical metric is defined by $G_{\alpha\beta} = c_s^{-2} g_{\alpha\beta} + (1 - c_s^{-2}) u_\alpha u_\beta$, whose inverse is $G^{\alpha\beta} = c_s^2 g^{\alpha\beta} - (1 - c_s^2) u^\alpha u^\beta$.

The characteristics associated with G are called sound cones.

Remark 6.5. We have already seen that $G_{\alpha\beta}$ is in fact a Lorentzian metric (provided that $|u|_g^2 = -1$, which is the case).

Definition 6.6. The null-forms relative to G are the following quadratic forms:

$$Q^{(G)}(\varphi, \psi) = G^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\psi,$$
$$Q_{\alpha\beta}(\varphi, \psi) = \partial_\alpha\varphi\partial_\beta\psi - \partial_\beta\varphi\partial_\alpha\psi.$$

The use of null-forms has a long history in hyperbolic PDEs and we will highlighted their properties below.

6.2. **The new formulation.** We can now state the new formulation of the relativistic Euler equations. As the actual statement of the new formulation is quite long, we will give only a schematic statement. We will use $\simeq$ to denote "up to harmless terms," where harmless have means from the point of view of the applications we discuss further below.

Theorem 6.7. *Assume that $(\widehat{h}, s, u)$ is a C^3 solution to the relativistic Euler equations. Then, $(\widehat{h}, s, u)$ also verify the following system of equations: Wave equations:*

$$\square_G \widehat{h} \simeq D + Q(\partial\widehat{h}, \partial_u) + L(\partial\widehat{h}),$$
$$\square_G u^\alpha \simeq C^\alpha + Q(\partial\widehat{h}, \partial_u) + L(\partial\widehat{h}, \partial_u),$$
$$\square_G s \simeq D + L(\partial\widehat{h}),$$

Transport equations:

$$u^\lambda\partial_\lambda s = 0,$$
$$u^\lambda\partial_\lambda S^\alpha \simeq L(\partial u),$$
$$u^\lambda\partial_\lambda \varphi^\alpha \simeq L(\partial\widehat{h}, \partial u).$$

Transport-div-curl equations:

$$u^\lambda \partial_\lambda D \simeq C + Q(\partial S, \partial \widehat{h}, \partial \omega) + L(\partial h, \partial u),$$
$$vort^\alpha(S) = 0,$$
$$\partial_\lambda \varphi^\lambda \simeq L(\partial \widehat{h}),$$
$$u^\lambda \partial_\lambda C^\alpha \simeq C + D + Q(\partial S, \partial \varphi, \partial \widehat{h}, \partial u) + L(\partial S, \partial \varphi, \partial \widehat{h}, \partial u).$$

Above, $L(\partial f_1, \cdots, \partial f_m)$ denotes linear combinations of terms that are at most linear in ∂f, whereas $Q(\partial f_1, \cdots, \partial f_m)$ denotes linear combinations of the null forms relative to G. $\Box_G$ is the wave operator w.r.t. G, and in $\Box_G u^\alpha$ the wave operator acts on u^α treated as a scalar function.

Proof. See [21]. □

One new result we can prove using the new formulation is that the entropy and n-orthogonal vorticity can be proven to one degree more regular than what is given by standard theory:

Theorem 6.8. *The relativistic Euler equations are locally well-posed (i.e., existence, uniqueness, and continues dependence on the data) with $(h, s, u, \varphi) \in H^N \times H^{N+1} \times H^N \times H^N, N > \frac{3}{2} + 1$.*

Proof. Proof: see [21]. □

Standard theory (e.g. symmetric hyperbolic systems or the mixed order formulation we derived earlier) gives only $(h, s, u, \varphi) \in H^N \times H^N \times H^N \times H^{N-1}$.

Remark 6.9. The above theorem assumes that the initial data enjoys the extra regularity $s(0) \in H^{N+1}$ and $\varphi(0) \in H^N$ (otherwise the result cannot be true since there is no smoothing in time for hyperbolic equations). The point is that standard theory gives $s \in H^N$ and $\varphi \in H^{N-1}$ even if such extra regularity for the data.

The above extra regularity ultimately comes from the div-curl part of the system. We point out, however, that this is not immediate as it may sound, since the div-curl system is for the spacetime div and curl, from which we need to extend three-dimensional regularity.

The extra regularity is an interesting result in itself, but it is in fact one of the important ingredients in the study of shocks in relativistic Euler, which we discuss next.

6.3. **The study of shock formulation.** Roughly, a shock wave, or shock for short, is a region in spacetime for which the solution remains bounded but one of its derivatives blows up.

While it is known that shocks can form for the relativistic Euler equations (see, e.g., [14]) for smooth initial data, we are interested in the problem of constructive proofs of stable formulation without symmetric assumptions in more than one spatial dimension, henceforth referred to simply as the problem of shock formulation, by which we mean:

Shocks form for an open set B of (small) initial data (usually perturbation of constant solutions). (Stability.)

contains "arbitrary" initial data, i.e., not restricted to a symmetry class

Proofs are constructive, so that we can get a precise description of the shock profile. (Needed for continuing the solution past the shock in a weak sense)

The framework needed to establish proofs of shock formulation involves the following ingredients:

Ingredient one: nonlinear geometric options. This is done by introducing an eihonel function U, which is a solution to the eihonal equation

$$G^{\alpha\beta} \partial_\alpha U \partial_\beta U = 0,$$

with appropriate initial condition. The eihonal function plays two crucial roles.

First, the level sets of U are the characteristics associated with the metric G, which are the sound cones. In this regard, we note that U is adapted to the wave part of the system and not to the transport part. This choice is based on the fact that the transport part corresponds to the evolution of the vorticity and entropy, and there are no kown blow-up results for these quantities. On the other hand, the only known mechanism of blow-up for relativistic Euler is the intersection of the sound cones. In particular, this shows the importance, in the context of shock formulation, of not treating the transport and sound part of the system together, as it is done in the first order symmetric hyperbolic formalism. The intersection of the sound cones is measured by the inverse foliation density μ defined as

$$\mu = -\frac{1}{G^{\alpha\beta}\partial_\alpha t \partial_\beta U},$$

which has the property that $\mu \to 0$ corresponds to the intersection of the characteristics.

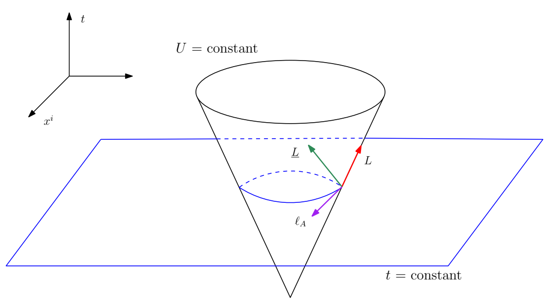
Second, in order to detect the blow up, we need to identify precisely in which direction(s) the solution blows up, and which direction it remains bounded. This is done with the introduction of a null-frame

$$\{e_1, e_2, \underline{L}, L\}$$

adopted to the sound cones. Here, L and $\underline{L}$ are null vectors, with respect to G, satisfying $G(\underline{L}, L) = -2$, and $\{e_1, e_2\}$ is an orthonormal, with respect to G, frame on the (topological) spheres given by the intersections

$$\{t = \text{constant}\} \cap \{U = \text{constant}\}.$$

We also have that $G(e_A, L) = 0 = G(e_A, \underline{L})$, $A = 1, 2$.



We can decompose quantities w.r.t. this null frame, and identify that blow-up occurs in the $\underline{L}$ direction, while derivatives of the fluid variables in the other directions remain bounded. To carry out the analysis, we also introduce a geometric system of coordinates adapted to the sound characteristics,

$$\left\{t, U, v^1, v^2\right\},$$

where v^A, $A = 1, 2$, are coordinates on the spheres $\{t = \text{constant}\} \cap \{U = \text{constant}\}$(they are constructed open solving $G^{\alpha\beta}\partial_\alpha U \partial_\beta v^A = 0$ with appropriate initial conditions0.

Ingredient two: nonlinear null-structure. The basic philosophy for the proof of shock formation is to show that, relative to the geometric coordinates $\{t, U, v^1, v^2\}$, the solution remains bounded all way to the shock. In this way, we transform the problem of shock formation into a more traditional one, where the goal is to derive long-time estimates for the solution (relative to the geometric coordinates). The blow-up of the solution w.r.t. the original coordinates is recovered by showing that the geometric coordinate system degenerates (in a precise fashion) relative to the original coordinates (since the characteristics are intersecting at the shock, we expect the geometric coordinates to degenerate there).

A crucial aspect of these constructions is that the null-frame and the geometric coordinates depend on the fluid's solution variables, since they (the null frame and the geometric coordinates) are constructed out of U which depends on G. (in broad philosophical terms, this resembles the approach to Einstein's equations, where the wave coordinates depend on the solution, i.e., on the spacetime metric). Therefore, in order to implement these ideas we have to show that the geometric coordinates remain regular all way up to the shocks. And to do so we need to obtain precise estimates for the fluid variables, showing, in particular, that the derivatives tangent to the sound cone do not produce singularities, the latter coming from derivatives in the $\underline{L}$ direction, as mentioned. In practice, this is done by showing that the dynamic can be decomposed into a Riccati-type term that drives the blow-up (recall that the Riccati ODE is $\frac{dz}{dt} = z^2$, which blows up in finites time) and error terms that do not significantly alter the high-frequency behavior of the Riccati term. Such terms appear as follows (we will illustrate with $\widehat{h}$, similar statements hold for u^α). Expanding the covariant wave operator to the null frame we find that the equation for $\widehat{h}$ reads, schematically,

$$L(\underline{L}\widehat{h}) \simeq -(\underline{L}\widehat{h})^2 + Q,$$

where Q denotes linear combination of null forms relative to G (and we omit harmless terms, e.g., terms linear in derivatives). The equation $L(\underline{L}\widehat{h}) \simeq -(\underline{L}\widehat{h})^2$ is the Riccati equation for the variable $\underline{L}\widehat{h}$, since L is differentiation in the direction of L, thus $L = \frac{d}{d\tau}$ for a suitable parametrization of the flow lines of L. Thus, we need to show that Q is a perturbation that does not significantly alter the Riccati behavior. This is problematic because Riccati forms are generally unstable under perturbations. However, and here is where the role of null-forms is important, Riccati terms are stable upon perturbations by null forms.Relative to the null-frame, we have

$$Q(\partial\varphi, \partial\psi) = \tau(\varphi)\psi + \tau(\psi)\partial\varphi,$$

where τ is differentiation tangent to the sound cones. This implies that even though Q is quadratic, it never involves terms quadratic in the direction the system wants to blow-up. Specifically, in our case, we then have

$$L(\underline{L}\widehat{h}) \simeq -(\underline{L}\widehat{h})^2 + \tau(\widehat{h})\partial h$$

so that the first term on the RHS is the only term quadratic in $\underline{L}1\widehat{h}$. If instead of $\tau(\widehat{h})$ we had $\partial\widehat{h}$ then we would get a $(\partial\widehat{h})^2$ term. After decomposing in a null frame, this $(\partial\widehat{h})^2$ could produce a $(\underline{L}\widehat{h})^2$ that cancels or nearly cancels the $-(\underline{L}\widehat{h})^2$ term from the Riccati part, thus working against the blow-up and preventing us from proving that shocks form. The term $\tau(\widehat{h})\partial\widehat{h}$, on the other hand, is at most linear in $\underline{L}\widehat{h}$ so that

$$L(\underline{L}\widehat{h}) = -(\underline{L}\widehat{h})^2 + \tau(\widehat{h})\underline{L}\widehat{h}.$$

Since the tangential derivatives remain bounded, the first term on the RHS dominates over the last term, leading to the blow-up of $\underline{L}\widehat{h}$.

Remark 6.10. A straw man ODE analogy of the above is the following. Consider the two following perturbations of the Riccati ODE $\frac{dz}{dt} = z^2 : \frac{dz}{dt} = z^2 + \varepsilon z, \frac{dz}{dt} = z^2 \pm \varepsilon z^3, z(0) > 0, \varepsilon > 0$ small. The first equation still blows up and it does it at the same rate as the original one. For the second perturbation, depending on the sign $\pm$ the solution will either exist for all time or it will blow up at an entirely different rate (thus effectively altering the blow-up). The null-forms are the PDE analog of the εy perturbation.

Ingredient three: energy estimates and regularity. The previous arguments assume that we can in fact close estimates establishing several elements needed in the above discussion (e.g. that tangential derivatives do in fact remain bounded). Thus, we need to derive estimates not only for the fluid variables but also for the eihonal function (since the regularity of the null-frame is tied to that of U).

Energy estimates for the fluid variables are obtained by commuting the equations with derivatives, but in order to avoid generating uncontrollable source terms, we need to commute the equations with certain vector fields that are adapted to the sound characteristics. This leads to vector fields of the form $z \sim \partial U \cdot \partial$. Commuting through, e.g., the equation for $\widehat{h}$:

$$\begin{aligned} z(\square_g \widehat{h}) &\sim \square_g (z\widehat{h}) + (\square_g \partial U)\partial \widehat{h} \\ &\sim \square_g (z\widehat{h}) + \partial^3 U \cdot \partial \widehat{h}. \end{aligned}$$

so the equation for $\widehat{h}$ gives

$$\square_g (z\widehat{h}) \sim \partial^3 U \cdot \partial \widehat{h} + \cdots$$

Since U solves a (fully non-linear) transport equation, standard regularity theory for transport equations gives that U is only as regular as the coefficients of the equation, which in this case is G, and since $G = G(\widehat{h}, s, u^\alpha)$, we find $\partial^3 U \sim \partial^3 G \sim \partial^3 \widehat{h} + \cdots$ On the other hand, standard energy estimates for wave equations give that form $\square_g (z\widehat{h})$ we obtain control of $\partial(z\widehat{h}) \sim \partial^2 \widehat{h}$, so in the end we are trying to control $\partial^2 \widehat{h}$ in terms of $\partial^3 \widehat{h}$ and thus have a derivative loss.

It turns out that we can overcome the regularity loss by exploiting some delicate tensorial properties of the eihonal equation and of the wave equation relative to geometric coordinates. Together these properties can be used to show that certain geometric tensors constructed out of U enjoy extra regularity in directions tangent to the sound cones. Carefully accounting for the precise structure of the aforementioned $\partial^2 U \partial h$ term we can show that it is precisely one of such terms with extra regularity. It turns out that all terms seem to exhibit loss of regularity are of this form and can thus be controlled.

The special structures mentioned above that are used to prevent loss of regularity of the eihonal function are tied to the geometry of the sound cones. The improved estimates, without regularity loss, for U are not based directly on the eihonal equation, but rather on evolution equations for geometric quantities (the null second foundamental form, mean curvature, etc.) of the sound cones.

To close the estimates we also need to use the extra regularity that we obtained for s and φ to close the estimates. To see this, let us do a naïve derivative counting. From the equation for u^α we have $\square_g u \sim C$, so we can control $\partial u \lesssim C$. But $C \sim \text{vort}^\alpha(\varphi) \sim \partial\omega$. From the transport equation for φ, $u^\lambda \partial_\lambda \varphi \sim \partial u$, we can control $\varphi \sim \partial u$, so in the end we are controlling $\partial u \partial^2 u$, which has a loss of a derivative. This loss of regularity can be avoided, however, by using the extra regularity for φ mentioned earlier. Something similar happens with some estimates involving s.

Finally, we mentioned that the energy estimates that are needed are in fact weighted estimates, where the weight is given by the inverse foliation density μ. Since $\mu \to 0$ at the shock, we end up with energies that are singular at top order. This is a major technical point that involves a complex bootstrap argument to close the estimates.

The above ingredients seem to be needed to establish proofs of shock formation, and are used in all known such proofs (in $n \geq 2$, see below). The crucial point for us here is that all such ingredients are present in the new formulation of the relativistic Euler equations.

6.4. **Some context for the work on shocks.** The ingredients outlined above have not all being introduced in [21]. They are the culmination of a series of beautiful ideas developed by a series of authors. For the sake of time we will not review this history here, but we refer to the introduction of CITATION HEREEEE.

When the fluid is irrotational, the new equations reduce significantly and agree with those found by Christodoulor [6] .The inclusion of vorticity causes several new difficulties and it is quite remarkable that the vorticity case presents many of the good structures found (and needed) in the irrotational case.

Finally, we mention that in one spatial dimension, the picture is compellingly simpler: in 1d we can rely essentially on the method of characteristics. While this is essentially the same as introducing an eihonal functional, in 1d we can dispense with all the geometric machinery discussed above. Also, we do not need to carry out energy estimates. Instead, one uses estimates in BV (bounded variation) spaces. It is possible to prove that such BV estimates do not generalize to two or more spatial dimensions [19].

7. Relativistic fluids with viscosity

So far we only discussed perfect relativistic fluids. There are important applications in physics where it is known that viscosity plays a key role. One such instance is in the study of the quark-gluon-plasma, an exotic type of matter, modeled as a fluid, that forms in heavy-ion collisions (such as those formed at the large Hadron Colliser). Another example is in the study of neutron star mergers. These are very active fields of research and we refer to [[4],[20]] for more discussion.

What is striking about the study of relativistic viscous fluids is that it is not settled what the convert equations are. There are several different models of relativistic viscous fluids in the literature. The abundance of models is due to the fact that, as it turns out, it is extremely difficult to construct models of relativistic viscous fluids that incorporate relevant physics and are causal and stable. (Causality is a fundamental postulate of relativity stating that no information propagates faster than the speed of light. Stability here means mode stability of the linearized equations).

For the sake of time, we will not discuss here the difficulties in constructing models of relativistic viscous fluids, nor will we review the several models available in the literature. We refer to [[3],[20]] for such discussions.

The first theory of relativistic viscous fluids that was showed to be causal and stable and to have a solution to the Cauchy problem coupled to Einstein's equations is the theory introduced in [3] (see [11] for the proofs).

Unfortunately, the model introduced in [3] is limited to conformal fluids for which, in particular, the equation of state is always $p = \frac{1}{3}\varrho$. Moreover, existence and uniqueness of solutions for this model has been established only in Gevrey Spaces, which are too restrictive for applications such as the numerical study of its equations.

Despite the existence of several different approaches to the problem, there exists one theory of relativistic fluids, the Mueller-Israel-Stewart(MIS) theory [9] that is widely used in physics. This is because the MIS has been used to construct successful models of the quark-gluon plasma. The MIS equations have been showed to be stable and to respect causality at the linearized level.

For the study of neutron star mergers, one needs to couple the fluid equations with Einstein's equations. It is not known whether the MIS can also be used to study neutron star mergers. This is because non-linearities are expected to play a major role in such mergers, and, as mentioned, only the linearized MIS equations have been proven to be causal (but see below).

Moreover, only recently, using state-of-the-art numerical simulations [1], it has become clear that viscous effects cannot be neglected in neutron star mergers. Interestingly, such simulations also indicate that it is bulk viscosity, as oppose to shear viscosity, that plays a major role in the mergers of neutron stars. It is sensible, therefore, to study the MIS equations with bulk viscosity and no shear viscosity, in which case the equations coupled to Einstein's equations, become:

$$R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta} = \overbrace{(p+\varrho)u_\alpha u_\beta + p g_{\alpha\beta}}^{energy\ monenton\ tencor\ of\ a\ perfect\ flvid} + \Pi \pi_{\alpha\beta}.$$
$$\nabla_\alpha (n u^\alpha) = 0.$$
$$\tau u^\alpha \nabla_\alpha \Pi + \Pi + \lambda \Pi^2 + \zeta \nabla_\alpha u^\alpha = 0.$$

Above, Π is a new variable incorporating the dynamics of bulk viscosity (in the MIS theory the viscous contributions are given by new variables, rather than by an expression in the velocity and density as in the classical Navier-Stokes equations). The last equation is the equation of motion for Π (since this new variable has been introduced, we need a new equation of motion as well), and τ and λ are known function of ϱ and u. It is also assumed that an equation of state $p = p(\varrho, u)$ is given and that $|u|_g^2 = -1$.

Theorem 7.1. *Under mild and physically reasonable assumptions, the Cauchy problem for Einstein's equations coupled to the MIS equations (only with bulh viscosity, as above) can be solved for initial data in Sobolev spaces. Moreover, the system is causal.*

Proof. see [4] for a precise statement and its proof. □

These is much more to be said about relativistic viscous fluids. This brief discussion is intended only as an illustration of the following fact: the study of relativistic viscous fluids is a very active area of research in physics. However, very little is known about the mathematical properties of models of relativistic fluids with viscosity and a great deal of basic, physically relevant, and important mathematical questions remain open.

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