Elementary realization of BRST symmetry and gauge fixing

Martin Rocek,
notes by Marcelo Disconzi

Abstract

This are notes from a talk given at Stony Brook University by Professor PhD Martin Rocek. I tried to write down as many details of the lecture as I could, although I may have missed some points (and so I am the only responsible for any imprecision which may be found).

1 General idea

We are going to illustrate the idea of BRST symmetry and gauge fixing with a very basic example using finite-dimensional integrals instead of path integrals.

Consider the following integral

$$\int dx\, e^{-S(r)}, \text{ where } x \in \mathbb{R}^2 \text{ and } r = |x| \tag{1}$$

We are supposing that the action S depends on r only, i.e, it is rotationally invariant. Therefore the action is invariant under the action of $SO(2)$, what can be expressed infinitesimally as

$$\delta S = 0 \tag{2}$$

for

$$\delta r = 0,\ \delta\theta = \epsilon \tag{3}$$

So states which are related to one another by a rotation are physically equivalent — and in this simplified example states are (labeled by) just points in the plane —, and therefore if we want to integrate over all (non-equivalent) physical states we should integrate only over points which are in different orbits. In this case the orbit of each point under the $SO(2)$ action is simply a circle. In other words, if we write the integral in polar coordinates

$$\int_0^\infty \int_0^2 \pi r\, dr\, d\theta\, e^{-S(r)} = 2\pi \int_0^\infty r dr e^{-S(r)} \tag{4}$$

we have that for each fixed r the integral over θ — which gives the factor 2π — is over-counting states. Therefore if want to get rid of this over-counting we need to drop the factor 2π. Stated in other terms, the integral over physical states is an integral on the quotient space or space of orbits. In the above example this can be written in terms of the original integral simply dividing by 2π:

$$I = \int_0^\infty r\, dr\, e^{-S(r)} \tag{5}$$

$$= \frac{1}{2\pi} \int dx\, e^{-S(r)} \tag{6}$$

Notice that the factor 2π that we need to divide by is exactly the volume of the symmetry group, and here lies the problem: in general the group of gauge transformations has infinite volume so this procedure does not work (of course, in this example we could guess that the desired integral is given by (5), but in general we start with some complicated path integral and we want to pass to an integral on the quotient, but we do not know how this integral on the quotient looks like and, as said, we can not obtain it by dividing by the volume of the group) . Therefore let us try to rewrite (5) in terms of (1) in a way which does not require division by the volume of the group.

We start by choosing a point in each orbit; this corresponds to choose a representative for each orbit. This can be accomplished by specifying a function $f(r, \theta) = 0$ which crosses each orbit transversally and only once; specifying f is a gauge-fixing. Now notice that I can be written as

$$I = \int dx e^{-S(r)} \delta(f(r, \theta)) \frac{\partial f}{\partial \theta} \tag{7}$$

Indeed

$$\int dx\, e^{-S(r)} \delta(f(r, \theta)) \frac{\partial f}{\partial \theta} = \int r\, dr\, d\theta\, e^{-S(r)} \delta(f(r, \theta)) \frac{\partial f}{\partial \theta}$$

$$= \int r\, dr\, df\, e^{-S(r)} \delta(f) = \int r\, dr\, e^{-S(r)} = I \tag{8}$$

(the term $\frac{\partial f}{\partial \theta}$ corresponds to what in the QFT setting is the Faddeev-Popov determinant) An example for f is $f(r,\theta) = \theta$. Since I is independent of f we can average over f's. In order to do this we use a Gaussian distribution:

$$\frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} dy \, e^{-\frac{1}{2\alpha}y^2} = 1 \tag{9}$$

So we can write

$$I = \frac{1}{\sqrt{2\pi\alpha}} \int dx \int dy \, e^{-S(r)} \delta(f(r,\theta) - y) e^{-\frac{1}{2\alpha}y^2} \frac{\partial f}{\partial \theta} \tag{10}$$

The term $\delta(f(r,\theta)-y)$ occurs because we set $f(r,\theta) = y$ instead of $f(r,\theta) = 0$ since we want to average over f's (obviously there is no lost of generality in doing this). Performing the integral in y:

$$I = \frac{1}{\sqrt{2\pi\alpha}} \int dx \, e^{-S(r) - \frac{1}{2\alpha}(f(r,\theta))^2} \frac{\partial f}{\partial \theta} \tag{11}$$

Our goal is to write everything as an argument of the exponential (so that the inegrand looks like $e^{-\text{action}}$), so we need to get rid of $\frac{\partial f}{\partial \theta}$. To do this we introduce anti-commuting or Grassman variables b, c, i.e., we consider supermanifolds. Recall the Berezian

$$\int dc \, c = 1, \quad \int dc = 0 \tag{12}$$

(analogous for b). Then

$$e^{-b\frac{\partial f}{\partial \theta}c} = 1 - b\frac{\partial f}{\partial \theta}c \tag{13}$$

since all higher order terms in the expansion of the exponential vanish because of the non-commutativity ($c^2 = b^2 = 0$). Then

$$\frac{\partial f}{\partial \theta} = \int db \, dc \, e^{-b\frac{\partial f}{\partial \theta}c} \tag{14}$$

Hence

$$I = \frac{1}{\sqrt{2\pi\alpha}} \int dx \, db \, dc \, e^{-S(r) - \frac{1}{2\alpha}(f(r,\theta))^2 - b\frac{\partial f}{\partial \theta}c} \tag{15}$$

It is convenient to introduce another variable (now it is a standard coordinate, not Grassman variables). We can rewrite I as

$$I = \frac{1}{\sqrt{2\pi\alpha}} \int dx\, db\, dc\, dB\; e^{-S(r)-\frac{\alpha}{2}B^2+iBf(r,\theta)-b\frac{\partial f}{\partial\theta}c} \tag{16}$$

To see that (16) is equivalent to (15), on (16) complete the square of B and perform the B integral.

Notice that with all this manipulations we did not change the value of I. Therefore we succeeded in writing the integral over physical states as integral of $e^{-\text{action}}$ without dividing by the volume of the group (the factor $\frac{1}{2\pi}$ appearing here comes from the Gaussian integral we introduced and has nothing to do with the volume of the group). But now we have an integral on $\mathbb{R}^{3|2}$ $(= x, B|b, c)$ instead of $\mathbb{R}^2$ (=only x).

Some terminology: b and c are called *ghosts*; more precisely c is a *ghost* and b an *anti-ghost*; B is called *auxiliary field.* We define a grading on the variables saying that b has degree -1, the physical variables (i.e, x) and B have degree zero and c has degree 1 (the reason for this grading is going to become clear in a moment). The degree of this grading is called *ghost number*, denoted $\mathfrak{gh}$. Notice that $\mathfrak{gh}(AB) = \mathfrak{gh}(A) + \mathfrak{gh}(B)$

Before introducing the BRST symmetry, let us make an important remark. Due to the rotational invariance of $S(r)$, the critical points of S are degenerated, so we can not apply perturbation theory around its critical points. Now the argument of the exponential on (16) has non-degenerated critical points, so perturbation theory applies. This is another reason for considering the original integral written as in (16) (and is one of the motivations for the BV formalism).

Now consider the following transformation, called *BRST transformation*

$$Qr = 0, \quad Q\theta = c, \quad Qc = 0 \tag{17}$$

$$Qb = iB, \quad QB = 0 \tag{18}$$

It easily follows that $Q^2 = 0$, so Q is a degree one derivation of the variables, where the grading was defined above. The definition of the BRST transformation is not arbitrary as it may look at first glance. Here is the prescription: starting with the infinitesimal symmetry replace the parameters of the transformation by anti-commuting variables. In our case, the symmetry is given by (3) and the parameter is ϵ, so ϵ is replaced by c. The transformations

$Qb = iB, QB = 0$ always hold independently of the group of transformations we start with. Qc needs to be figured out in each specific example.

Since S depends on r only and $Qr = 0$ we have $QS = 0$. Notice also that

$$Q(\frac{\alpha}{2} iBb + -bf) = -\frac{\alpha}{2}B^2 + iBf - b\frac{\partial f}{\partial \theta}c \tag{19}$$

(we used $Qf(r,\theta) = c\frac{\partial f}{\partial \theta}$). Then (16) can be rewritten as

$$I = \frac{1}{\sqrt{2\pi\alpha}} \int d\mu \; e^{-S(r)+Q\psi} \tag{20}$$

where $d\mu = dx\, db\, dc\, dB$ and $\psi = -\frac{\alpha}{2}B^2 + iBf - b\frac{\partial f}{\partial \theta}c$. Since $QS = 0$ and $Q^2 = 0$ we see that the BRST transformation is a symmetry of the "new action" $S + Q\psi$.

Because we have a grading and a derivation we can consider the cohomology of this complex, called *BRST cohomology*. Notice that terms of the form $Q\phi$ do not contribute to the integral. Recall that the physical variables have zero degree; $Qr = 0$ tells us that the relevant physical variable (recall that the integral over physical states does not depend on θ) is a cocycle. Therefore the "observables" are given by the 0^{th}-cohomology group (notice that the other 0^{th} degree variable is a coboundary: $B = -iQb$, so it does not play any role in the cohomology).

Before moving to another example, let us remark that (7) is really everything we need in order to compute the integral over physical states without dividing by the volume of the group, but the remaining manipulations were convenient in order to make the integral more doable.

As another example, consider now

$$I = \int dx \; e^{-S(x^1)} \tag{21}$$

As in the case $dr\, d\theta$, here we are integrating over two variables $dx = dx^1 dx^2$ but the action depends only on one of them: $S = S(x^1)$. So S is invariant under translations along the x^2 direction and any two states related by such a translation are physically equivalent. So an integral over different physical states is given by

$$I = \int_{-\infty}^{\infty} dx^1 \; e^{-S(x^1)} \tag{22}$$

But here, differently from the previous example, we can not obtain (22) from (21) by dividing by the volume of the group of symmetries, since the group of translations along x^2 is isomorphic to $\mathbb{R}$ and hence has infinite volume. However, the prescription using the BRST transformation works since it makes no reference to the volume of the group.

Following the prescription of BRST symmetry, first we write the infinitesimal transformation for the x^2-translation:

$$\delta x^1 = 0,\ \delta x^2 = \lambda \tag{23}$$

So the BRST transformation is

$$Qx^1 = 0, \quad Qx^2 = c, \quad Qc = 0 \tag{24}$$

$$Qb = iB, \quad QB = 0 \tag{25}$$

Then proceeding as before we get

$$I = \frac{1}{\sqrt{2\pi\alpha}} \int d\mu\, e^{-S(x^1)+Q\psi} \tag{26}$$

1.1 Application to Gauge fields

Now we want to apply this idea to path integrals. We *assume* that everything we did before carries over to this infinite dimensional setting. Consider as example an Yang-Mills action. Then applying the above procedure the integral over physical states reads

$$I = \int [DA] e^{-S(A)+Q\psi} \tag{27}$$

More precisely, we have the following data: let M be the affine space of Lie algebra valued 1-forms on a four-dimensional manifold (space-time) X, G the gauge group (e.g. $SU(2)$) and $\mathcal{G}$ the group of gauge transformations. The connections transform as

$$d + A \mapsto g^{-1}(d + A)g,\ g = g(x), x \in X, A \in M \tag{28}$$

The action is

$$\frac{1}{4\lambda}|F|^2 + \frac{\theta}{2\pi} F \wedge F \tag{29}$$

and the BRST symmetry

$$AQ = dc + [A, c], \; Qc = \frac{1}{2}[c, c] \tag{30}$$

$$Qb = iB, \; Qb = 0 \tag{31}$$

(here even though we have some isotropy we do not care about it since it is finite dimensional and we are in an infinite dimensional setting). It turns out that

$$\psi = i\frac{\xi}{2}Tr(bB) + Tr(b \star (d + A) \wedge \star A) - \frac{\xi}{2}TrB^2 + Tr(B \star (d + A) \wedge \star A)$$
$$+Tr(b \star (d + A) \wedge dc \star \wedge (dc + [A, c])) \tag{32}$$

$$S_{BV} = S + Tr(A^*(dc + [A, c]) + \frac{1}{2}c^*[c, c]) \tag{33}$$

where ξ is a parameter.

2 BV formalism

In order to introduce the BV formalism, let us consider the following example. Suppose we have a rotationally invariant action in $\mathbb{R}^3$; the symmetry group is $SO(3)$. It is useful to keep in mind the very first example we developed, of a rotationally invariant action on the plane. Recall also that the gauge fixing is given by a function which select a representative in each orbit (the orbits here are spheres centered at the origin). We can do this by choosing a direction n on space and putting $f(x) = n \times x, \; x \in \mathbb{R}^3$; $\times$ is the cross product of vectors. So the gauge fixing condition is $f(x) = 0$, i.e.,

$$n \times x = 0 \tag{34}$$

This is the exact analogous of what we did on the plane: choosing the direction n corresponds to choose a line from the origin which crosses each sphere (=each orbit) once[1]; in the two dimensional case we choose a line from the origin which crossed each circle (there the orbits were circles) once by setting $f(r, \theta) = \theta = 0$.

[1] Actually, $n \times x = 0$ gives a line through the origin which crosses each sphere *twice* but for doing perturbation theory this really does not matter

Now write the infinitesimal symmetry as

$$\delta x = x \times \lambda \tag{35}$$

$\lambda = (\lambda^1, \lambda^2, \lambda^3)$ since the $SO(3)$ has three parameters. Therefore we will have three ghosts and three anti-ghosts (recall that each parameter of the transformation gives rise to a ghost), so the BRST transformation gives:

$$Qx = x \times c \tag{36}$$

But now we have a problem: the BRST symmetry also "contains a symmetry":

$$Q(x + \gamma x) = x \times (c + \gamma x) = x \times c \tag{37}$$

where γ is a parameter. It follows then that (as it would be expected in case we have a symmetry) $S + Q\psi$ is a degenerate quadric form, so we can not apply perturbation theory around a critical point. Indeed, this becomes explicit if we write $Q\psi$:

$$\psi = b \cdot (n \times x) + \frac{\alpha}{2} b \cdot B \tag{38}$$

$$Q\psi = B \cdot (n \times x) + \frac{\alpha}{2} B \cdot B + b \cdot (n \times (x \times c)) \tag{39}$$

The term $b \cdot (n \times (x \times c))$ is the "bad term" in ψ for it makes the form degenerated due to the mentioned symmetry $c \mapsto \gamma$. It should be noticed that this symmetry stems from the fact that the isotropy subgroup of $SO(3)$ of an arbitrary point on $\mathbb{R}^3$ is not trivial.

The BV formalism is developed to overcome this difficulty (although it applies even when we do not have such "symmetry of the symmetry"). The idea is to treat all the variables — physical ones, auxiliary fields, ghosts and anti-ghosts — on equal footing, and apply the ideas used before, i.e., add extra variables in order to make the quadratic form non-degenerate. In doing this, we are going to have to add extra variables in order to eliminate the symmetry of $Qx = x \times c$. When we first added new variables in order to deal with the symmetry of the action we called these variables ghosts. Since now c *is* a ghost, the new variables which eliminate that symmetry are called *ghosts-for-ghosts*; these will be responsible for fixing this extra symmetry that we found.

Now we proceed as follows: first let us develop the ideas of the BV formalism using the first example as a guide, then we go back to the $SO(3)$ example and deal with it.

2.1 BV algebra

The basic idea is to double all the fields. A bit more formally, we are going to introduce an odd sympletic structure (or supersympletic structure). Consider the symmetry

$$\delta x^i = \lambda \epsilon^{ij} x^j, \text{ i.e. } \delta x^2 = \lambda x^1, \ \delta x^2 = -\lambda x^1 \tag{40}$$

(we are writing the infinitesimal rotations in Cartesian coordinates, differently from what we did at the begining when we used polar coordinates) Then we have

$$I = \frac{1}{2\pi} \int_{\mathbb{R}^{3|2}} d^2x \, dB \, db \, dc \, e^{-S-Q\psi} \tag{41}$$

$$\psi = -\frac{\alpha}{2} iB + bf(x^1, x^2) \tag{42}$$

$$Qx^1 = cx^2, \quad Qx^2 = -cx^1, \quad Qc = 0 \tag{43}$$

$$Qb = iB, \quad QB = 0 \tag{44}$$

Let ϕ denote any one of the fields, i.e., $\phi \in \{x^1, x^2, c, b, B\}$. Add fields ϕ^* with opposite statistics. So $\phi \in \mathbb{R}^{3|2}$ and $\phi^* \in \mathbb{R}^{2|3}$. ϕ^* are called *anti-fields* (do not confuse anti-fields with anti-ghosts). If ϕ has ghost number n then ϕ^* has ghost number $-n-1$.

Define the bracket

$$(A, B) = \frac{\overrightarrow{\partial} A}{\partial \phi} \frac{\overleftarrow{\partial} B}{\partial \phi^*} - \frac{\overleftarrow{\partial} A}{\partial \phi^*} \frac{\overrightarrow{\partial} B}{\partial \phi} \tag{45}$$

(a sum over fields and corresponding anti-fields is understood here), where the arrow $\rightarrow$ means derivative "from the left" and $\leftarrow$ means derivative "from the right", e.g.

$$\frac{\overrightarrow{\partial}}{\partial c}(bc) = -b, \ \frac{\overleftarrow{\partial}}{\partial c}(bc) = b \tag{46}$$

since b and c anti-commute. If follows

$$(\phi, \phi) = (\phi^*, \phi^*) = 0, \ (\phi, \phi^*) = 1 \tag{47}$$

Now we look for an action S_{BV} such that

$$Q\phi = (S_{BV}, \phi) \tag{48}$$

The idea is that S_{BV} will encode information about both Q and S. We construct S_{BV} of the form $S_{BV} = S + S_{min} + S_{nm}$, were *min* stands for *minimal* and *nm* for *non-minimal*. The non-minimal part contains the fields b and B and its corresponding anti-fields, the minimal part contains the remaining ones. In our example, equation (48) is satisfied if we put

$$S_{min} = (x^1)^* c x^2 - (x^2)^* c x^1 \tag{49}$$

$$S_{nm} = -ib^* B \tag{50}$$

For example (recall that ϕ^* has the opposite statistics):

$$(-ib^* B, b) = iB = Qb \tag{51}$$

$$((x^1)^* c x^2, x^1) = c x^2 = Q x^1 \tag{52}$$

The non-minimal part is always given by (50), so the interesting part is the minimal one.

Now we have

$$Q\phi = (S_{BV}, \phi) \Rightarrow Q^2 \phi = (S_{BV}, (S_{BV}, \phi)) = \frac{1}{2}((S_{BV}, S_{BV}), \phi) \tag{53}$$

$$\text{hence } (S_{BV}, S_{BV}) = 0 \text{ implies } Q^2 = 0 \tag{54}$$

$(S_{BV}, S_{BV}) = 0$ is called *master equation*. Notice that if we set all anti-fields to zero we recover S, i.e.,

$$S_{BV}|_{\phi^*=0} = S \tag{55}$$

Let us introduce the operator

$$\Delta = \frac{\overrightarrow{\partial}}{\partial \phi} \frac{\overleftarrow{\partial}}{\partial \phi^*} \tag{56}$$

Then

$$\Delta e^{-\frac{\Gamma}{\hbar}} = 0 \Leftrightarrow \Delta \Gamma - \frac{1}{2\hbar}(\Gamma, \Gamma) = 0 \tag{57}$$

Indeed, apply Δ to $e^{-\frac{\Gamma}{\hbar}}$, use the chain rule and the identity

$$\begin{aligned} (\Gamma, \Gamma) &= \frac{\overrightarrow{\partial}\Gamma}{\partial \phi} \frac{\overleftarrow{\partial}\Gamma}{\partial \phi^*} - \frac{\overleftarrow{\partial}\Gamma}{\partial \phi^*} \frac{\overrightarrow{\partial}\Gamma}{\partial \phi} \\ &= \frac{\overrightarrow{\partial}\Gamma}{\partial \phi} \frac{\overleftarrow{\partial}\Gamma}{\partial \phi^*} + \frac{\overrightarrow{\partial}\Gamma}{\partial \phi} \frac{\overleftarrow{\partial}\Gamma}{\partial \phi^*} = 2 \frac{\overrightarrow{\partial}\Gamma}{\partial \phi} \frac{\overleftarrow{\partial}\Gamma}{\partial \phi^*} \end{aligned} \tag{58}$$

Here $\Gamma = \Gamma(\phi, \phi^*)$ is a function of ϕ and ϕ^* (in general Γ is thought of as a quantum version of S_{BV} and (57) as a quantum master equation). $\Delta\Gamma$ is called *anomaly*. We want solutions to

$$\Delta\Gamma = 0 \tag{59}$$

$$(\Gamma, \Gamma) = 0 \tag{60}$$

(in general in the mathematical literature people treat the full equation $\Delta\Gamma - \frac{1}{2\hbar}(\Gamma, \Gamma) = 0$).

Now we can rewrite the gauge fixing Lagrangian $S + Q\psi$ as $[S_{BV} + (S_{BV}, \psi)]|_{\phi^*=0}$. Define

$$e^{\mathcal{L}_\psi} S_{BV}|_{\phi^*=0} = [S_{BV} + (S_{BV}, \psi) + \frac{1}{2}((S_{BV}, \psi), \psi) + \dots]|_{\phi^*=0} \tag{61}$$

In most examples higher order terms vanish if ψ does not contain anti-fields and ψ is linear in the anti-fields. Then

$$e^{\mathcal{L}_\psi} S_{BV}|_{\phi^*=0} = [S_{BV} + (S_{BV}, \psi)]|_{\phi^*=0} = S + Q\psi \tag{62}$$

2.2 $SO(3)$ symmetry

Now we return to the example. Recall that we have

$$\psi = b \cdot (n \times x) + \frac{\alpha}{2} b \cdot B \tag{63}$$

$$Q\psi = B \cdot (n \times x) + \frac{\alpha}{2} B \cdot B + b \cdot (n \times (x \times c)) \tag{64}$$

and $S + Q\psi$ is degenerate. We want to construct $S_{BV} = S + S_{min} + S_{nn}$ and a new ψ which satisfies the master equation. Recall that S_{nm} is always the "trivial" part, so what we really need is S_{min}. The idea for "guessing" how to construct these quantities is that we should add as many anti-fields are necessary to simultaneously (i) make ψ non-degenerate and (ii) make S_{BV} to satisfy the master equation. Moreover, we want ψ to involve the least possible number of fields and anti-fields and S_{min} the largest possible. Keeping in mind that ψ has ghost number -1 and S_{BV} has ghost number 0 we can see which terms are allowed to be added (for example, we can't add a ghost number 0 term to ψ). Looking at the grading wee see which new variables we need to introduce:

$$
\begin{array}{ccccccccc}
-2 & & -1 & & 0 & & 1 & & 2 \\
 & & & & x & & & & \\
 & & & \swarrow & & \searrow & & & \\
 & & b & & & & c & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
\beta & & & & B, \alpha & & & & \gamma \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & \\
 & & \mathbb{B} & & & & A & &
\end{array}
\tag{65}
$$

The left-to-right arrows indicate the action of Q; the right-to-left indicate the action of what would be called "anti-Q" (which we do not treat here). The new variable γ was introduced in order to eliminate the aforementioned symmetry Q, so it is an example of what is called ghosts-for-ghosts. The ghost-for-ghost has degree or ghost number 2, one above the degree of the ghost (which is 1). If we had a situation where γ also has a symmetry, then we would need to introduce more variables to kill that symmetry; and we would end up with a degree 3 variable i.e., a "ghosts-for-ghosts-for-ghosts". Notice that γ is a scalar since the isotropy subgroup is $U(1)$ (one-dimensional) (or, said differently, because the parameter appearing on (37) is one-dimensional). The anti-fields A and $\mathbb{B}$ were also introduced.

Now we can write down what we need:

$$\psi = b \cdot (n \times x) + \frac{\alpha}{2} b \cdot B + \alpha \mathbb{B} + \beta A + \beta n \cdot c + \alpha n \cdot b \tag{66}$$

$$S_{nm} = -i b^* \cdot B + \beta^* \cdot \mathbb{B} + \alpha A \tag{67}$$

$$S_{min} = x^* \cdot (x \times c) - \frac{1}{2} c^* \cdot (c \times c) + c^* \cdot x \gamma \tag{68}$$

(notice that the degrees match, for example, the ghost number $\mathfrak{gh}$ of βA is $\mathfrak{gh}(\beta) + \mathfrak{gh}(A) = -2 + 1 = -1 = \mathfrak{gh}(\psi)$).

Now it is a lengthly calculation to show that all our requirements are fulfilled: ψ is non-degenerate and S_{BV} satisfies the master equation, $(S_{BV}, \phi) = Q\phi$; this is left as an exercise.