Spontaneous symmetry breaking

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1 Introduction

The demand that the Lagrangian be gauge invariant does not allow gauge fields to have a mass term in the Lagrangian. In other words gauge invariance seems to imply that gauge fields are massless, because[1] $m^2 A_a A^a$ is not gauge invariant [2] .

However, in nature the carrier of exchange particles $W^\pm, Z$-bosons have been observed to have a mass (these are carrier particles for the weak force between particles). On the other hand we feel very strongly that all background forces should be described by gauge fields. Spontaneous symmetry breaking is a method by which these massive bosons can be treated under the context of gauge theory, i.e., a way around adding the mass term $m^2 A_a A^a$ to the Lagrangian which keeps gauge invariance. This is a key step in the electroweak theory of Salam-Weinberg.

Before presenting the mechanism of symmetry breaking, we first briefly review some facts about exchanging particles.

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[1]The reason why this is called the mass term comes from generalizing the Klein-Gordon equation. Recall that $m^2 = E^2 - |\vec{p}|^2$ from Special Relativity (with $c = 1$). Upon quantizing: $|\vec{p}|^2 \mapsto (-i\nabla)^2$ and $E^2 \mapsto H^2$, where H is the Hamiltonian (with $\hbar = 1$). But Schroedinger's equation says that $i\partial_t \psi = H\psi$. Thus this equation in quantized form becomes $m^2 \psi = (i\partial_t)^2 \psi - (i\nabla)^2 \psi$, i.e, $\Box \psi - m^2 \psi = 0$. So in the Lagrangian $m^2 \psi^2$ is interpreted as the mass term.

[2]To be more precise, we do know how to simply add gauge invariant mass terms (not of the form $m^2 A_a A^a$, of course), but these turn out to be highly non-linear and in four dimensions lead to non-renormalizable theories. The Higgs mechanism produces a mass term which leads to a "less non-linear" (hence more treatable) and renormalizable theory.

2 Massive and massless particles

Let us show first that in general short range forces are massive ($W^\pm$, Z-bosons for the weak force) while long range forces should be massless (photon for electromagnetism (EM) and graviton for gravity). The gluons of the strong force are massless even though it is short range, so the idea "short range $\leftrightarrow$ massive, long range $\leftrightarrow$ massless" is just an approximation (but a good one, this idea leads to the discovery of the pion which approximately describes the strong force, see below). To proceed let us recall that the Heisenberg uncertainty principle: $\Delta t \Delta E \geq \frac{\hbar}{2}$. This means that we need a certain amount of time to make an accurate measurement of energy. It follows that we can briefly violate conservation of energy without violation of quantum mechanics (QM). For us to be able to tell the difference between two energy levels E_a and E_b, we must be able to measure energy to an accuracy ΔE smaller than the difference $E_a - E_b$, i.e., $\Delta E \leq E_a - E_b$. But to measure energy to this accuracy we must have a certain amount of time $\Delta t \geq \frac{\hbar}{2(E_a - E_b)}$, i.e., the energy state must last for at least this period of time. So, if it does not last at least at least $\frac{\hbar}{2(E_a - E_b)}$, then we will not notice violations of conservation of energy. For example, suppose we have an electron at rest. According to Einstein $E_a = m_e c^2$ in this state. If this electron emits a photon the energy of this new state is given by

$$E_b = \frac{m_e c^2}{\sqrt{1 - \frac{v_e^2}{c^2}}} = m_e c^2 + K_e + E_{\text{photon}}$$
$$= m_e c^2 + K_e + \hbar\nu > E_a$$

where we have used that the energy of the photon equals $\hbar\nu$, $E_b > E_a$ implies that energy is not conserved. Here we assume that the electron recoils when it emits the photon so that it now has a velocity v_e, this preserves conservation of momentum. Even though classically this can not happen, this would not violate QM if this photon is absorbed by another electron before the amount of time $\Delta t \geq \frac{\hbar}{2(E_a - E_b)}$ has elapsed. The absorbing electron will also recoil in the opposite direction to preserve momentum. The Feynman diagram showed in figure 1.

As long as the total energy-momentum before equals the total energy-momentum after — so, as long as the four-momentum is conserved —, this does not violate QM, what can happen if Δt is small. Conclusion: the
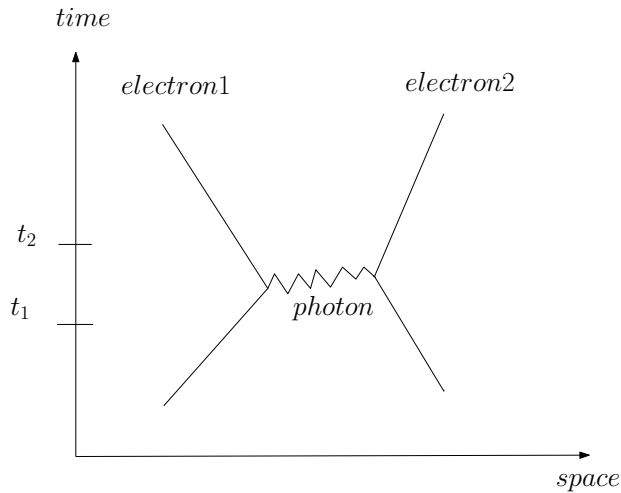
Figure 1: Electromagnetic force.

electrons have exchanged a virtual photon[3] and repelled each other. This explains electromagnetism!

Roughly all forces should act like this[4]. Suppose we have a force moderated by a massive particle, then we can get an estimate on the range of the force. If a carrier particle of mass m is exchanged, by the uncertainty principle the largest it can last without violate QM is

$$\Delta t \leq \frac{\hbar}{2\Delta E} = \frac{\hbar}{\frac{2mc^2}{\sqrt{1-\frac{v^2}{c^2}}}} \leq \frac{\hbar}{2mc^2}$$

If we let this particle travel at nearly the speed of light, the furthest it could travel is $\Delta r = c\Delta t \leq \frac{\hbar}{2mc}$. Thus for infinite range like gravity and EM, the mass of their exchange particles should be zero.

The weak force has a range about 10^{-18} m. Thus the mass of the weak force exchange particle should be $\frac{\hbar}{2c\Delta r} \approx 10^{-25}kg$. Indeed, these particles

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[3]Of course, when carrier particles are detected by experiments they happen to be always on-shell. Although this may sound weird, it is a typical quantum effect, similar to that of the double slit: the electron diffracts forming and interference pattern on the wall, but as soon as we put a detector to know by which slit it passed the interference pattern disappears.

[4]the arguments outlined here help to understand the idea short range $\leftrightarrow$ massive, long range $\leftrightarrow$ massless, but they also give orders of magnitude close to the precise values (see below).

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have a mass of approximately $10^{-25}kg$[5]. Therefore this very simple reasoning leads to a rough approximation of the masses of the carriers of the weak force. The electroweak theory predicted the $W^{\pm}$ and Z bosons with the correct masses and indeed they were discovered in 1983 at CERN. The discoverers (of course!) won the Nobel Prize.

But even before the discovery of $W^{\pm}$ and Z bosons, Yukawa used arguments along these lines: we know the range of the strong force between protons and neutrons, it is about 10^{-15} meters. Thus the carrier particle should have mass around

$$m = \frac{\hbar}{2c\Delta r} \approx 10^{-27}kg$$

In 1925 Yukawa predicted this particle, and in 1947 the pion was discovered with mass similar to the above value . Yukawa and Powell (the experimenter who found it) shared the Nobel Prize for this discovery.

But this argument does not work for the strong force. Indeed, Yukawa theory ran into a lot of difficulties in late 1950's and early 1960's, for example, at very high energies the strong force could not be explained by pion exchange. In 1964 the quark model was proposed and in this model the pion is a bound state of a quark-antiquark pair. The strong force is now understood by the exchange of gluons between quarks. Moreover, the pion is made up of quarks and that explain why it is an approximation for carrier particles of the strong force! The gluons have been discovered (the first direct experimental evidence was in 1979) and are massless, but the strong force between quarks acts in a very different way to all other forces: it gets stronger as distance increases.

3   Back to spontaneous symmetry breaking

A QM system can be in various energy states given by $H\psi_n = E_n\psi$. The state of minimum energy E_0 (not necessarily zero) is called the *vacuum or ground state*. If a single vacuum state corresponds to E_0 it is said to be *non-degenerate* (like for eigenvectors) and *degenerate* otherwise.

Suppose the system is invariant under some internal symmetry group G. The vacuum is said to be *invariant under G* if it is transformed into itself by the action of G; *non-invariant* otherwise. If the vacuum state and the

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[5]$140.35 \times 10^{-27}kg$ the W boson and $160.81 \times 10^{-27}kg$ the Z boson.

Lagrangian are invariant under G we say the system has an *exact symmetry*[6]. If the vacuum state is not invariant but the Lagrangian is, then the symmetry is said to be *spontaneously broken* (or equivalently we say that the system has a *spontaneously broken symmetry*). The crucial point (to be developed in examples below) is that in order to construct a theory we must choose one of the various vacua at hand. Although there is a symmetry relating the vacua, we "break the symmetry" by specifying a preferred vacuum state. Finally, if the vacuum state and the Lagrangian are not invariant then this is called an *explicit symmetry breaking.*

As a very intuitive example think of a pen which is put in upright position so carefully so that it does not fall under the influence of gravity, i.e., the pen is in an equilibrium "state". Clearly this is very symmetric state: it is invariant under rotations around the axis which contains the pen. However, this equilibrium is also very unstable. Under any small perturbation the pen will fall and reach a new equilibrium state. Now this new equilibrium state, despite being stable, is no longer invariant under rotations, so the system "has lost symmetry" or the symmetry is "broken".

4   Exact symmetry

We now consider several examples. The first is exact symmetry. Consider the complex $\lambda\phi^4$ Lagrangian

$$\mathcal{L} = \overline{\partial_a \phi}\partial^a \phi - m^2 |\phi|^2 - \lambda |\phi|^4$$

where λ is a constant called *coupling constant* and m is the mass of the field ϕ (although coupling constants play an important role in perturbation theory, here the reader can think of λ simply as a term which must be introduced in order to make the last term to have the correct units). If $\lambda \neq 0$ then the Euler-Lagrange (EL) equations are non-linear and the system is *interacting*[7]. We may consider $m^2|\phi|^2 + \lambda|\phi|^4$ as a potential energy term. This Lagrangian comes up as a simplified model for the Higgs boson, which will be discussed

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[6]It is worthwhile to mention that by a result of Coleman, known as Coleman's theorem, if the vacuum state is invariant under G so is the Lagrangian.

[7]When the EL are linear the principle of superposition of solutions holds, so waves add linearly. In other words, there is no scattering, so the theory is *free*; if the EL are non-linear there will be scattering, this is why the theory is then said to be interacting.

later. The EL equations are

$$\Box \phi + m^2 \phi + 2\lambda \phi |\phi|^2 = 0$$

This Lagrangian is clearly invariant under the $U(1)$ symmetry $\phi' = e^{i\theta}\phi$ (global symmetry). By calculating the stress energy-tensor we know what the energy for this field is:

$$E(t) = \int_{\mathbb{R}^3} |\partial_t \phi|^2 + \overline{\nabla \phi} \cdot \nabla \phi + m^2 |\phi|^2 + \lambda |\phi|^4$$

The integrand is the *energy density*; usually we denote the energy density by the same letter and call it simply "energy" (although a bit confusing, this is a common practice, analogous to calling the Lagrangian density simply by Lagrangian). Since all terms are positive, the minimum energy is $E_0 = 0$, and the only vacuum state is $\phi(x) \equiv 0$. Notice that this trivially satisfies the EL equations — indeed, if we assume spatial homogeneity, which is the case in most models, the vacuum will always satisfy EL. The vacuum is non-degenerate since $e^{i\theta}(0,0) = (0,0)$, showing that the vacuum is transformed into itself under the action of the group.

5 Spontaneous breaking of global symmetry

Now consider the opposite sign for the m^2 term, i.e,

$$\mathcal{L} = \overline{\partial_a \phi} \partial^a \phi + m^2 |\phi|^2 - \lambda |\phi|^4$$

The Lagrangian is still invariant under the global $U(1)$ symmetry[8]. Now the energy density is

$$E = |\partial_t \phi|^2 + \overline{\nabla \phi} \cdot \nabla \phi - m^2 |\phi|^2 + \lambda |\phi|^4 \tag{1}$$

(the total energy is the integral of the above quantity). It is clear that the minimum of (1) is obtained when $|\partial_t \phi|^2 = \nabla \phi = 0$, i.e, $\phi = constant$ and

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[8]Recall that a symmetry is global if its parameters are constants, e.g., $\phi' = e^{i\theta}\phi$ with θ=constant. The symmetry is local if we allow the parameters to depend on the coordinates, e.g., $\phi' = e^{i\theta(x)}\phi$. The process of going from global to local symmetry by allowing the parameters to depend on coordinates is referred as *localizing* or *gauging* the symmetry and it is the starting point of gauge theories.

$V(\phi) = -m^2|\phi|^2 + \lambda|\phi|^4$ is at a minimum (this is the same thing as finding the minimum for the function $f(x) = -m^2x^2 + \lambda x^4$). The critical points are at $\phi = 0$ and $|\phi|^2 = \frac{m^2}{2\lambda}$. But clearly the global minimum is at $|\phi|^2 = \frac{m^2}{2\lambda}$. This can be view from the graph of $V(\phi)$ (a "Mexican hat", see figure 2). Thus, we get a one-parameter family of ground states, the circle in the complex plane given by $|\phi|^2 = \frac{m^2}{2\lambda}$ (as before, note that this solves the EL equations).
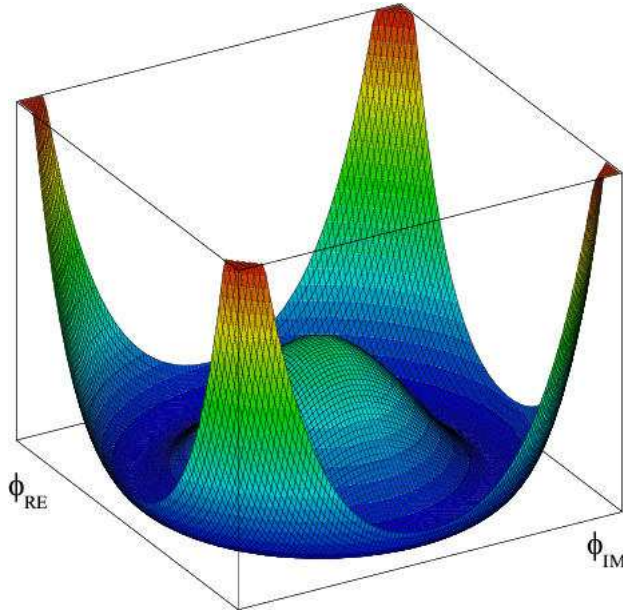

Figure 2: Mexican hat potential.

Therefore, we have spontaneous symmetry breaking. Indeed, These vacuum states are degenerate; the $U(1)$ symmetry transformation $\phi' = e^{i\theta}\phi$ rotates these vacua along the circle. Each gets transformed into another but never into itself (unless, of course, $\theta = 0$).

Because the vacuum value is $|\phi(x)| = \sqrt{\frac{m^2}{2\lambda}}$, plugging this value into (1) we find that the minimum energy (density) is not zero. We can shift this expectation value in order to make it zero by changing the Lagrangian $\mathcal{L}' = \mathcal{L} - V(|\phi|^2 = \frac{m^2}{2\lambda}) = \mathcal{L} + \frac{m^4}{4\lambda}$; this does not change the physics (for example, the EL equations are the same) but now $E' = E + \frac{m^4}{4\lambda} = 0$.

In order to construct a theory, a definite stable vacuum — what in our example corresponds to a point on the circle $|\phi|^2 = \frac{m^2}{2\lambda}$ — must be chosen. To different vacuum states correspond different theories. This can be explained

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because the probability of tunneling between two minima is zero. To see this, consider the following heuristic argument from [CN]. The probability of tunneling between two minima decreases with increasing the number of degrees of freedom, and for an infinite number of degrees of freedom (which is the case for a field) this probability vanishes. In fact, for a field in a finite volume Ω the Lagrangian of a system is $\mathcal{L}' \sim \int d^3x \mathcal{L} \sim \mathcal{L}\Omega$, the kinetic energy is $\sim |\partial_t \phi|^2 \Omega$ and the potential energy is $\sim V\Omega$ where $V = |\nabla\phi|^2 - m^2|\phi|^2 + \lambda|\phi|^2$. The problem reduces to calculating the quantum-mechanical probability of passing a barrier of width $\sim \sqrt{\frac{m^2}{2\lambda}}$ and height $\sim \frac{\Omega m^4}{\lambda}$ by a particle of mass $\sim m\Omega$. This probability is proportional to $\exp(-\frac{\Omega m^3}{\lambda})$ and tens to zero as $\Omega \to \infty$, i.e., transitions between two vacuum states are not possible. Intuitively, we are saying the following: if $V(\phi)$ is the energy *density* for the field then the potential energy of the system is $\sim \Omega V(\phi)$. Therefor in order to transfer the field from one ground state to another we have to add energy proportional to $\Omega V(\phi)$, what would require infinity energy when $\Omega \to \infty$ (see [Ru] for more details).

Since any vacuum state can be transformed into another by a change of gauge, they can all be chosen on equal footing. It is customary to choose the one on the real axis, i.e., $\phi_0 = \sqrt{\frac{m^2}{2\lambda}}$, and build the theory upon this vacuum. Small perturbations of the vacuum are *excited states* and are interpreted quantum mechanically as particles, so we consider small perturbations around ϕ_0, that is, $\phi(x) = \phi_0(x) + \phi_1(x) + i\phi_2(x)$, where ϕ_1 and ϕ_2 are real. Plugging this into the original Lagrangian yields

$$\mathcal{L}_1(\phi_1, \phi_2) = \partial_a(\phi_0 + \phi_1 - i\phi_2)\partial^a(\phi_0 + \phi_1 + i\phi_2)$$
$$+m^2(\phi_0 + \phi_1 - i\phi_2)(\phi_0 + \phi_1 + \phi_2) - \lambda[(\phi_0 + \phi_1 - i\phi_2)(\phi_0 + \phi_1 + i\phi_2)]^2$$
$$= \partial_a\phi_1\partial^a\phi_1 - 2m^2\phi_1^2 + \partial_a\phi_2\partial^a\phi_2 + \mathcal{L}_2$$

where

$$\mathcal{L}_2(\phi_1, \phi_2) = -\lambda(\phi_1^2 + \phi_2^2)^2 - 2m^2\phi_1(\phi_1^2 + \phi_2^2) + \frac{m^2}{2}\phi_0^2$$

is called the *interaction Lagrangian*. The term $\frac{m^2}{2}\phi_0^2$ is just a constant and can be removed without changing the physics.

We conclude the following: spontaneous symmetry breaking of global symmetry has lead to an equivalent theory (the Lagrangians $\mathcal{L}$ and $\mathcal{L}_2$ are equivalent, both describe the dynamics of the system [CN]) of a massive

real scalar field ϕ_1 and a massless real scalar field ϕ_2 called a *Goldstone boson* (sometimes Nambu-Goldstone boson)[9]. Of course, we know that actual symmetry in particle physics is described by local symmetry. We will see that in that case Goldstones also appear, but only at an intermediate step; they are *not* physical particles. Instead of interpreting Goldstones as physical fields, it is more accurate to interpret them as parameterizing the orbit $|\phi|^2 = \frac{m^2}{2\lambda}$. In other words, we know that in gauge theory the physical states are the orbits (=equivalent classes of gauge related states) and a Goldstone is a representative of such orbit.

Remark: The term $m^2|\phi|^2$ in the original Lagrangian is not a mass term since it has the wrong sign and indicates imaginary mass. To find the true mass spectrum we had to break the symmetry to get the real scalar field ϕ_1 with a mass term of correct sign. Thus ϕ_1 is the massive particle associated to this theory and there is also a massless one, the Goldstone (see page 42 of Chaichian/Nelipa).

6   Spontaneous symmetry breaking of local symmetry and Higgs mechanism

In this section we develop examples spontaneous symmetry breaking of local symmetry.

6.1   Abelian theory

Consider EM coupled to a scalar field; the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} + \overline{\nabla_a\phi}\nabla^a\phi + m^2|\phi|^2 - \lambda|\phi|^4$$

where as usual $\nabla_a\phi = \partial_a - ieA_a\phi$ and $F_{ab} = \partial_a A_b - \partial_b A_a$. This Lagrangian is invariant under the *local* $U(1)$ symmetry. The total energy of the system (which can be found with the stress-energy tensor) is

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}\sum_{j=1}^{3} F_{0j}^2 + \sum_{j<k} F_{jk}^2 + |\nabla_0\phi|^2 + \sum_{j=1}^{3}|\nabla_j\phi|^2 - m^2|\phi|^2 + \lambda|\phi|^4$$

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[9]There is a surprising general fact, known as Gladstone's theorem, which states that spontaneous symmetry breaking of *global* symmetry always leads massless particles; all such massless particles are known as Goldstones bosons.

(recall that the first and second sum in the integrand are the norm square of the electric and magnetic fields E and B). The ground state must have $E = B = 0$, what implies that A_a is a closed and exact one-form, i.e., the vector potential is what physicists call a "pure gauge": $A_a = \frac{1}{e}\partial_a\alpha$ for some α. We must also have $\nabla_a\phi = 0$ so that $\partial_a\phi - i(\partial_a\alpha)\phi = 0$, what gives $\phi(x) = e^{i\alpha(x)}\phi_0$ where as before $|\phi_0|^2 = \frac{m^2}{2\lambda}$. Therefore $|\phi(x)|^2 = \frac{m^2}{2\lambda}$ in order to minimize the potential term. Note that since the energy functional is gauge invariant, if (A_a, ϕ) is a ground state then so is $(A_a + \frac{1}{e}\partial_a\alpha, e^{i\alpha})$ for any α. We must choose one to build the theory, so we set $(A_a^0, \phi_0) = (0, \frac{m}{\sqrt{2\lambda}})$. Since the vacuum state is not invariant under local $U(1)$ transformations but the Lagrangian is, we have that the system presents spontaneous symmetry breaking.

Now we consider perturbations around the ground state: $(A_a, \phi_0 + \phi_1 + i\phi_2)$. Putting this into the Lagrangian produces

$$\mathcal{L}_1(\phi_1, \phi_2, A_a) = -\frac{1}{4}F_{ab}F^{ab}$$

$$+[\partial_a(\phi_0 + \phi_1 - i\phi_2) + ieA_a(\phi_0 + \phi_1 - i\phi_2)][\partial^a(\phi_0 + \phi_1 + i\phi_2) - ieA^a(\phi_0 + \phi_1 + i\phi_2)]$$

$$+m^2(\phi_0 + \phi_1 - i\phi_2)(\phi_0 + \phi_1 + i\phi_2) - \lambda[(\phi_0 + \phi_1 - i\phi_2)(\phi_0 + \phi_1 + \phi_2)]^2$$

$$= \frac{1}{4}F_{ab}F^{ab} + \frac{e^2m^2}{2\lambda}A_aA^a + \partial_a\phi_1\partial^a\phi_1 - 2m^2\phi_1^2 + \partial_a\phi_2\partial^a\phi_2$$

$$-\frac{2me}{\sqrt{2\lambda}}A_a\partial^a\phi_2 + \mathcal{L}_2(\phi_1, \phi_2, A_a)$$

where $\mathcal{L}_2$ is the interaction Lagrangian which consists of terms of order greater than quadratic. Thus the vector fields has acquired a mass, and there are now massless and massive scalar fields ϕ_1, ϕ_2. But now the free part of the Lagrangian is not in canonical form — i.e., sum of quadratic parts of Lagrangians of different fields — due to the term $A_a\partial^a\phi_2$, but this can be removed by setting $B_a = A_a - \frac{1}{e\phi_0}\partial_a\phi_2$ so that

$$\mathcal{L}_1(\phi_1, \phi_2, B_a) = -\frac{1}{4}(\partial_a B_b - \partial_b B_a)(\partial^a B^b - \partial^b B^a) + e^2\phi_0^2 B_a B^a + \partial_a\phi_1\partial^a\phi_1$$

$$-2m^2\phi_1^2 + \mathcal{L}_2(\phi_1, \phi_2, B_a)$$

Now we only have a massive vector field and a massive scalar field.

Therefore, we have the following conclusion: the vector field A_a has "eaten" a Goldstone boson ϕ_2 (unphysical field) and acquired a mass. This

is the *Higgs mechanism* for giving mass to gauge fields. ϕ_1 is called the *Higgs boson* and is a physical fields which has yet to be observed. But now we need to find how the physical fields ϕ_1 and B_a transform. Under the gauge transformation $\phi' = e^{i\alpha}\phi$, $A'_a = A_a + \frac{1}{e}\partial_a\alpha$ we have, of course, that $\mathcal{L}_1$ is invariant; using $\phi' = \phi_0 + \phi'_1 + i\phi'_2 = e^{i\alpha}(\phi_0 + \phi_1 + i\phi_2)$ we obtain

$$\phi'_1 = (\phi_0 + \phi_1)\cos\alpha - \phi_2\sin\alpha - \phi_0$$
$$\phi'_2 = (\phi_0 + \phi_1)\sin\alpha + \phi_2\cos\alpha$$
$$B'_a = A'_a - \frac{1}{e\phi_0}\partial_a\phi'_2$$
$$= A_a + \frac{1}{e}\partial_a\alpha - \frac{1}{e\phi_0}\partial_a[(\phi_0 + \phi_1)\sin\alpha + \alpha_2\cos\alpha]$$
$$= A_a - \frac{1}{e\phi_0}(\partial_a\phi_2)\cos\alpha + \frac{1}{e\phi_0}\sin\alpha\partial_a\alpha\phi_2$$
$$+\frac{1}{e}\partial_a\alpha(1 - \cos\alpha) - \frac{1}{e\phi_0}(\cos\alpha)(\partial_a\alpha)\phi_1 - \frac{1}{e\phi_0}(\partial_a\phi_1)\sin\alpha$$

Notice that when considering infinitesimal gauge transformations — i.e. α is small and $\phi' = \phi + i\alpha\phi$ and $A'_a = A_a + \frac{1}{e}\partial_a\alpha$ —, we have $\phi'_1 = \phi_1$ and $B'_a = B_a$ since all other terms are of order $O(\alpha)$. Thus in the free Lagrangian portion of $\mathcal{L}_1$ only the gauge invariant fields ϕ_1, B_a remain.

Also we should get rid of the Goldstones in $\mathcal{L}_2$, the interaction part. This is done by choosing a gauge such that $\phi'_2 = 0$, what can be done by choosing α appropriately (this is equivalent to defining a new field ψ_1 to be a particular combination of ϕ_1, ϕ_2 and A_a or B_a). But notice that after that is no longer gauge invariant (since we are *choosing* a gauge [CN]), but then only the physical fields ϕ_1, B_a remain.

Remark: The Euler-Lagrange equations for these fields are

$$\partial_a B^{ab} + e^2\phi_0^2 B^b = 0$$
$$\Box\phi_1 + 2m^2\phi_1 = 0$$

6.2   Nonabelian case

Consider the Lagrangian for a pair of self-interacting Klein-Gordon fields

$$\mathcal{L} = (\partial_a\phi)^H\partial^a\phi + m^2|\phi|^2 - \lambda|\phi|^4$$

where $\phi = (\phi_1, \phi_2) \in \mathbb{C}^2$ and $|\phi|^2 = \phi^H \phi = |\phi_1|^2 + |\phi_2|^2$. This is clearly $SU(2)$ invariant and we can localize this symmetry by introducing a Yang-Mills field (i.e., a connection), so the new Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{ab}^\ell F^{\ell ab} + (\nabla_a \phi)^H (\nabla^a \phi) + m^2 |\phi|^2 - \lambda |\phi|^4$$

with the standard notation for Yang-Mills fields. For convenience of the reader we recall some definitions:

$$A_a(x) = \sum_{j=1}^{3} A_a^\ell(x) \sigma_\ell \in \mathfrak{su}(2), \; \sigma_\ell \text{ are generators for the algebra, i.e.:}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\nabla_a \phi = \partial_a \phi - ig A_a \phi$$

$$F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b]$$

$$F_{ab} = F_{ab}^\ell \sigma_\ell$$

i.e., $F = DA$ where D is the exterior covariant derivative; we can also write:
$F_{ab}^\ell = \partial_a A_b^\ell - \partial_b A_a^\ell + C_{km\ell} A_a^k A_b^m$ where the structure constants are given by

$$[\sigma_k, \sigma_m] = C_{km\ell} \sigma_\ell$$

The gauge transformation is

$$A_a' = \omega \partial_a \omega^{-1} + \omega A_a \omega^{-1}$$

$$\phi' = \omega \phi$$

$$F_{ab}' = \omega F_{ab} \omega^{-1}$$

$$\text{where } \omega(x) \in SU(2)$$

The Euler-Lagrange equations are

$$\nabla_a F^{ab} = J^b \text{ or } \partial_a F^{ab} - [A_a, F^{ab}] = J_\ell^b \sigma_\ell$$

$$\Box \phi + m^2 \phi - 2\lambda |\phi|^2 \phi = 0$$

The current is given by

$$J_\ell^b = i[(\nabla_b \phi)^H \sigma_\ell \phi - \phi^H \sigma_\ell \nabla_b \phi]$$

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and the energy density by

$$E = \frac{1}{2} \sum_i \sum_\ell F_{0i}^\ell F_{0i}^\ell + \frac{1}{2} \sum_{k<j} \sum_\ell F_{kj}^\ell F_{kj}^\ell + |\nabla_0 \phi|^2 + \sum_{j=1}^3 |\nabla_j \phi|^2$$

So in the ground state $F_{ij}^\ell = 0$ which implies that the gauge field is a pure gauge $A_a = \frac{i}{g} \omega \partial_a \omega^{-1}$. We must also have $\nabla_a \phi = 0$, what gives $\phi(x) = \omega(x)\phi_{cons}$ (integrate by parts to see that this solves the previous equation), where $|\phi_{cons}|^2 = \frac{m^2}{2\lambda}$. To choose a vacuum state to build the theory we take $A_a = 0$ and $\phi = (0, \phi_0)$ with $\phi_0 = \frac{m}{\sqrt{2\lambda}}$. As usual we look at perturbations around the vacuum. These can be conveniently written as:

$$\phi(x) = [\phi_0 + \chi_0(x)\sigma_0 + i\sigma_\ell \sigma_\ell] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \chi_2 + i\chi_1 \\ \phi_0 + \chi_0 - i\chi_3 \end{pmatrix}$$

(σ_0 is the identity) The fields χ_1, χ_2, χ_3 will be Goldstones while χ_0 is the massive Higgs field. Plugging this into the original Lagrangian yields

$$\mathcal{L}(\chi_0, \chi_1, \chi_2, \chi_3, A_a) = -\frac{1}{4}(\partial_a A_b^\ell - \partial_b A_a^\ell)(\partial^a A^{\ell b} - \partial^b A^{\ell a})$$

$$+ \frac{g^2 m^2}{2\lambda} A_a^\ell A^{\ell a} + \partial_a \chi_0 \partial^a \chi_0 - 2m^2 \chi_0^2 + \partial_a \chi_\ell \partial^a \chi_\ell - \frac{gm}{\sqrt{2\lambda}} A_a^\ell \partial^a \chi_\ell$$

$$+ \mathcal{L}_2(\chi_0, \chi_1, \chi_2, \chi_3, A_a)$$

where $\mathcal{L}_2$ is the interaction Lagrangian consisting of terms of at least cubic order. This appears to give three massless Goldstones χ_1, χ_2 and χ_3, a massive Higgs field χ_0 and three massive gauge fields A_a^1, A_a^2 and A_a^3. As before, we let the gauge fields "eat"' the Goldstones to acquire mass by setting $B_a^\ell = A_a^\ell - \frac{1}{g\phi_o} \partial_a \chi_\ell$, then

$$\mathcal{L}_1 = -\frac{1}{4}(\partial_a B_b^\ell - \partial_b B_a^\ell)(\partial^a B^{\ell b} - \partial^b B^{\ell a}) + g^2 \phi_0^2 B_a^\ell B^{\ell a} + \partial_a \chi_0 \partial^a \chi_0 - 2m^2 \chi_0^2$$

$$+ \mathcal{L}_2(\chi_0, \chi_1, \chi_2, \chi_3, B_a)$$

Conclusion: The Higgs mechanism for nonabelian gauge groups produces N massive gauge fields (they acquire a mass) if the group is N dimensional and one massive scalar (neutral, no charge) scalar field.

Remark: Again, one should choose a gauge such that $\chi_1' = \chi_2' = \chi_3' = 0$ so that all Goldstones from $\mathcal{L}_2$ vanish as well.

7 Electroweak theory of Glashow-Salam-Weinberg

The electroweak theory describes EM and the weak force. It illustrates (or is based on) partial breaking of gauge symmetry and the Higgs mechanism. The weak interaction governs β-decay in which nucleus expels an electron to get to a lower energy configuration. This is radiation: expeltion or emission of elementary particles from nuclei because of quantum fluctuations or instability of the configuration. α-decay is when a helim nucleus is emitted as radiation; γ-decay is when a photon is emitted as radiation.

In β-decay the weak interaction converts a neutron n^0 into a proton p^+ (which gives a radioactive isotope of the element) while emitting an electron e^- and an anti-neutrino $\overline{\nu_e}$ [10]:

$$n^0 \to p^+ + e^- + \overline{\nu_e}$$

This is due to the conversion of a down quark to an up quark (in the neutron) by emission of a W boson which subsequently decays into an electron and anti-neutrino (see figure 3).

As primary particles for this theory one can take the electron e^-, the μ^- lepton, the τ^- lepton and their neutrinos. Experimental evidence suggests that each par particle/neutrino should have an $SU(2)$ symmetry and thus should be considered as a doublet $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$, $\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}$ and $\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$. Since the μ^- and τ^- leptons are treated similarly to the electron, we restrict our attention to the electron. Of course e^- is represented by a 4-component Dirac field and so is the neutrino ν_e. The doublet $\psi = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ is then a 8-component object of 2 Dirac spinors. The Lagrangian is

$$\mathcal{L} = \psi^H \hat{\partial\!\!\!/} \psi + M^2 |\psi|^2$$

where

$$\hat{\partial\!\!\!/} = i\hat{\gamma}^a \partial_a, \quad \hat{\gamma}^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix}$$

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[10]Corresponding to each massive particle is an antiparticle with same mass but opposite charge. Even electrically neutral particles such as a neutron are not identical to their anti-particles, e.g., the neutron is made of quarks and the anti-neutron of anti-quarks. Neutrinos are elementary particles that travel close to the speed of light, lack an electric charge, pass through ordinary matter and have a tiny mass and are usually created from certain radioactive decay; they interact with gravity and the weak force but not with EM and the strong force.
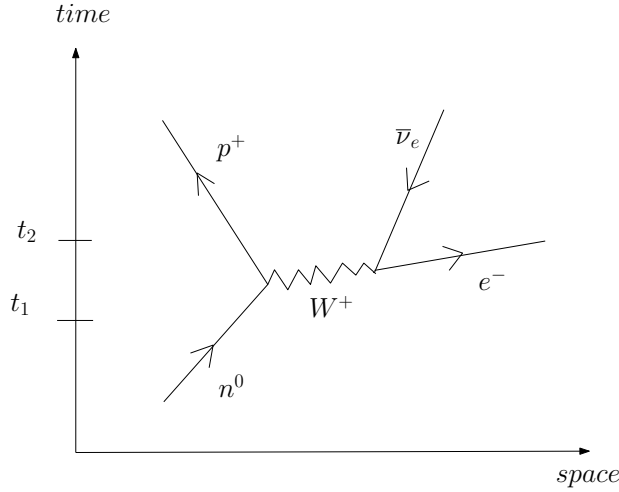
Figure 3: Electromagnetic force.

This clearly has a $SU(2)$ symmetry and a $U(1)$ symmetry. The total group is $SU(2) \times U(1)$ and the generators for $SU(2)$ give the weak interaction and for $U(1)$ gives the EM interaction.

Experimental evidence suggests that the exchange particles for the weak interaction have a mass so a Higgs field (massive scalar field) is needed to import a mass to those gauge fields. We must then add to the action the Lagrangian for the Higgs field. Since there are three generators which need to eat three Goldstones, we need a 2 component complex scalar field $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2$ to get the Higgs mechanism (three real fields will be eaten by the Goldstones and the remaining one will be the massive Higgs). The Lagrangian is now

$$\mathcal{L} = \psi^H \partial\!\!\!/ \psi + M^2 |\psi|^2 + (\partial_a \phi)^H \partial^a \phi + m^2 |\phi|^2 - \lambda |\phi|^4$$

The potential for the Higgs field and the coupling constant λ need to be determined experimentally.

Now we need to gauge these symmetries so the Lagrangian becomes

$$\mathcal{L} = \psi^H \nabla\!\!\!\!/ \psi + M^2 |\psi|^2 + (\nabla_a \phi)^H \nabla^a \phi + m^2 |\phi|^2 - \lambda |\phi|^4 - \frac{1}{4} F^\ell_{ab} F^{\ell ab} - \frac{1}{4} B_{ab} B^{ab}$$

where $B_{ab} = \partial_a B_b - \partial_b B_a$ is for $U(1)$ and $F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b]$ is for $SU(2)$ and $\nabla_a \phi = \partial_a \phi - ig A_a \phi - ie B_a \phi$ with $A_a = A^\ell_a \sigma_\ell$; also $\nabla\!\!\!\!/ = i\gamma^a \nabla_a$.

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Here g and e are coupling constants. An arbitrary gauge transformation is given by $\phi'(x) = \omega(x)e^{i\alpha}\phi = e^{i\theta_\ell\sigma_\ell + i\alpha\sigma_0}\phi$.

We now proceed as before. A similar calculation shows that the ground state occurs if $A_a = \omega\partial_a\omega^{-1}$, $B_a = \partial_a\alpha$ are pure gauges and $\nabla_a\phi = 0$. This gives $\phi = \omega e^{i\alpha}\phi_{const}$ with $|\phi_{const}|^2 = \frac{m^2}{2\lambda}$. We choose as usual $A_a = B_a = 0$, $\phi_{const} = (0, \phi_0)$, $\phi_0 = \frac{m}{\sqrt{2\lambda}}$. The perturbations around the ground state may be written as

$$\phi(x) = [(\phi_0 + \chi_0)\sigma_0 + i\sigma_\ell\chi_\ell(x)]\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \chi_2 + i\chi_1 \\ \phi_0 + \chi_0 - i\chi_3 \end{pmatrix}$$

Plugging into the Lagrangian yields (without the electron, i.e., we restrict to the bosonic sector of the theory from now on):

$$\mathcal{L} = -\frac{1}{4}(\partial_a A_b^\ell - \partial_b A_a^\ell)(\partial^a A^{\ell b} - \partial^b A^\ell a)\partial_a\chi_0\partial^a\chi_0 - 2m^2\chi_0^2$$

$$-\frac{1}{4}(\partial_a B_b - \partial_b B_a)(\partial^a B^b - \partial^b B^a) + \frac{g^2 m^2}{2\lambda}A_a^\ell A^{\ell a} + \frac{e^2 m^2}{2\lambda}B_a B^a$$

$$+\partial_a\chi_\ell\partial^a\chi_\ell - \frac{gm}{\sqrt{2\lambda}}A_a^\ell\partial^a\chi_\ell - \frac{gem^2}{\lambda}B_a(A_a^3 - \frac{1}{g\phi_0}\partial_a\chi_3) + \mathcal{L}_1$$

where $\mathcal{L}_1$ is the interaction Lagrangian consisting of terms of order greater or equal to three. Letting the vector fields eat the Goldstones by $\hat{A}_a^\ell = A_a^\ell - \frac{g\phi_0}{\partial a}\chi_\ell$ produces

$$\mathcal{L} = -\frac{1}{4}(\partial_a \hat{A}_b^\ell - \partial_b \hat{A}_a^\ell)(\partial^a \hat{A}^{\ell b} - \partial^b \hat{A}^\ell a)$$

$$+\frac{e^2 m^2}{2\lambda}B_a B^a - \frac{1}{4}(\partial_a B_b - \partial_b B_a)(\partial^a B^b - \partial^b B^a)$$

$$+\frac{g^2 m^2}{2\lambda}\hat{A}_a^\ell \hat{A}^{\ell a} + \partial_a\chi_0\partial^a\chi_0 - 2m^2\chi_0^2 - \frac{gem^2}{\lambda}B_a \hat{A}_a^3 + \mathcal{L}_1$$

The new problem here is the mixed term $B_a\hat{A}_a^3$.which prevents us from determining the mases of B_a, $\hat{A}_a^3$. To get rid of it we define

$$W_a^\pm = \frac{1}{\sqrt{2}}(\hat{A}_a^1 \pm i\hat{A}_a^2)$$

$$Z_a = \hat{A}_a^3 \cos\theta - B_a \sin\theta$$

$$Y_a = \hat{A}_a^3 \sin\theta + B_a \cos\theta$$

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where θ is a parameter to be determined. Note that the inverse transformation is

$$\hat{A}_a^1 = \frac{1}{\sqrt{2}}(\hat{W}_a^+ + W_a^-)$$

$$\hat{A}_a^2 = \frac{1}{i\sqrt{2}}(W_a^+ - W_a^-)$$

$$\hat{A}_a^3 = Z_a \cos\theta + Y_a \sin\theta$$

$$B_a = Y_a \cos\theta - Z_a \sin\theta$$

Compute:

$$\frac{g^2 m^2}{2\lambda} \hat{A}_a^3 \hat{A}^{3a} + \frac{e^2 m^2}{2\lambda} B_a B^a - \frac{gem}{\lambda} B_a \hat{A}^{3a}$$

$$= \frac{m^2}{\lambda}[(\frac{1}{2} g^2 \cos^2\theta + \frac{1}{2} e^2 \sin^2\theta + eg \cos\theta \sin\theta) Z_a Z^a +$$

$$(\frac{1}{2} g^2 \sin^2\theta + \frac{1}{2} e^2 \cos^2\theta - eg \cos\theta \sin\theta) Y_a Y^a +$$

$$+ (\frac{1}{2}(g^2 - e^2) \sin 2\theta - ge \cos 2\theta) Y_a Z^a]$$

So we choose θ such that the mixed term vanishes: $\frac{1}{2}(g^2 - e^2)\sin 2\theta - ge \cos 2\theta = 0$. The parameter θ is called *mixing angle* and of course depends on the coupling constants. More precisely (using $\sin 2\theta = 2\sin\theta \cos\theta$ and $\sin 2\theta = \cos^2\theta - \sin^2\theta$):

$$\cos\theta = \frac{g}{\sqrt{g^2 + e^2}} \quad \sin\theta = \frac{e}{\sqrt{e^2 + g^2}}$$

Notice that it follows that the coefficient in front of $Y_a Y^a$ also vanishes. The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}(\partial_a W_b^- - \partial_b W_a^-)(\partial^a W^{+b} - \partial^b W^{+a}) + \frac{g^2 m^2}{\lambda} W_a^- W^{+a}$$

$$-\frac{1}{4}(\partial_a Z_b - \partial_b Z_a)(\partial^a Z^b - \partial^b Z^a) + \frac{m^2(g^2 + e^2)}{2\lambda} Z_a Z^a$$

$$-\frac{1}{4}(\partial_a Y_b - \partial_b Y^a)(\partial^a Y^b - \partial^b Y^a) + \partial_a \chi_0 \partial^a \chi_0 - 2m^2 \chi_0^2 + \mathcal{L}_1$$

Since $W_a^- = \overline{W_a^+}$ this corresponds to a charge massive vector field; Z_a gives an uncharged massive vector field and Y_a a massless vector field; finally χ_0

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is a massive scalar field. These are the only physical fields, .i.e., only these combinations of the original A_a^ℓ and B_a are physically meaningful. χ_0 is of course the Higgs boson (which has yet to be detected) and naturally we interpret Y_a as the gauge field for the electromagnetism.

One can raise the following question: why did one vector field, Y_a, not acquire a mass, but the others did?

The answer is that the $SU(3) \times SU(2) \times U(1)$ symmetry was only partially broken. To see this we claim that the ground state remains invariant under the group $\hat{U}(1)$ with generator $\sigma_0 + \sigma_3$. Since

$$e^{i\alpha(\sigma_0+\sigma_3)}\phi_{cons} = \begin{pmatrix} e^{2i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} = \phi_{cons}$$

Thus the ground state is invariant and so is the Lagrangian, hence this symmetry is exact and is not broken. The other three generators are broken. Note that this subgroup $\hat{U}(1)$ is not the standard $U(1)$ factor of $SU(2) \times U(1)$. This indicates that the EM potential should be a linear combination of B_a and $\hat{A}_a^3$ since the group is generated by $\sigma_0 + \sigma_3$. The other three generators which are broken and corresponding to $W_a^\pm$ and Z_a are $\frac{1}{\sqrt{2}}(\sigma_1 \pm \sigma_2)$, $g\sigma_3 - e\sigma_0$ or $\sigma_3 - \sigma_0$.

This was an example of the Higgs mechanism with partially broken gauge symmetry. *The number of massive vector fields corresponds to the number of broken generators.* This is sometimes referred to as *residual symmetry.*

Under an infinitesimal $\hat{U}(1)$ transformation $\phi' = e^{i\alpha(\sigma_0+\sigma_3)}\phi$ with α small, the gauge fields transform as

$$Z_a' = Z_a$$
$$\chi_0' = \chi_0$$
$$W_a'^\pm = W_a^\pm \pm i\alpha W_a^\pm$$
$$Y_a' = Y_a + \frac{1}{e}\partial_a\alpha$$

which is another good reason for Y_a to be identified with the EM field. Clearly the Lagrangian is invariant under this symmetry.

Remark: As always, to get rid of the Goldstones in the interaction term $\mathcal{L}_1$ we should choose the gauge $\chi_1' = \chi_2' = \chi_3' = 0$, so that only physical terms remain.

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