Some algebraic structures in physics

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Abstract

These are notes from a series of informal meetings the authors organized with the goal of sharing their different background in physics and mathematics. This partially explains some lack of coherence in the exposition.

1 The conformal group

Let (M, g) be a d-dimensional pseudo-Riemannian manifold. X is a conformal Killing vector field if $\mathcal{L}_X g = \lambda g$ for some non-negative funcion $\lambda : M \to \mathbb{R}$. We can compute (in local coordinates):

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\mu\rho}$$

If we replace the above partial derivatives by covariant derivatives the symmetry of the Christoffel symbols with respect to the lower indeces guarantee that the introduced extra terms cancel out, so we also have

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \nabla_\rho g_{\mu\nu} + \nabla_\mu X^\rho g_{\rho\nu} + \nabla_\nu X^\rho g_{\mu\rho}$$

Now recall that $\nabla g = 0$, so the first term above vanishes $g_{\mu\nu}$ is a constant for the covariant derivative, so we can pass the metric over the covariant derivative and lower the indices of X, obtaining

$$(\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$$

or $(\mathcal{L}_X g)_{\mu\nu} = X_{\nu;\mu} + X_{\mu;\nu}$ in phycisist's notation. So the condition of being conformal Killing can be expresses as

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \lambda g_{\mu\nu}$$

Applying this equations for the case of commutator of two vector fields it is possible to show that conformal Killing fields form a Lie algebra.

Contracting the above equation with $g^{\mu\nu}$ we obtain

$$\lambda = \frac{2}{d}\nabla_\rho X^\rho$$

hence

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \frac{2}{d} g_{\mu\nu}\nabla_\rho X^\rho \tag{1.1}$$

If $X_1, \dots, X_n$ are conformal Killing vector fields then exponentiating we get a Lie group, the *conformal group*. This is analogous to the situation where we have isometries: the Killing fields form the Lie algebra of the isometry grouop, so the isometries can be obtained by exponentiating the Killing fields. Notice also that by taking $\lambda = 0$ we recover the condition for X to be a Killing field, so the isometries are a subgroup of the conformal group.

1.1 Example.

Consider $\mathbb{R}^{p,q}$, i.e., $R^d, d = p+q$ with a metric η of signature $(---\cdots+++)$ and assume $d > 2$. Then eq. 1.1 can be written as

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho X^\rho \equiv \frac{2}{d}\eta_{\mu\nu}(\partial \cdot X) \tag{1.2}$$

Apply $\partial_\rho \partial_\kappa$ in the above equation, contract it with $\eta^{\rho\nu}$ to get

$$\partial_\mu \partial_\kappa (\partial \cdot X) + \Delta(\partial_\kappa X_\mu) = \frac{2}{d}\partial_\mu \partial_\kappa (\partial \cdot X)$$

Using ν instead of κ we can rewrite it as

$$d\Delta\partial_\nu X_\mu + (d-2)\partial_\mu\partial_\nu(\partial \cdot X) = 0$$

Now change the roles of μ and ν to get a similar equation, add the two equations and use 1.2 to get

$$\Big(\eta_{\mu\nu}\Delta + (d-2)\partial_\mu\partial_\nu\Big)(\partial \cdot X) = 0$$

This equation is second order and linear in $\partial \cdot X$, so X is at most quadratic in coordinates. The above equation gives ansatz for the original equation 1.2. Using this we can actually solve 1.2; the solutions are

1. $X^\mu = a^\mu$ (translations).
2. $X^\mu = \omega^\mu_\nu x^\nu$ (rotations), where x^ν are coordinates.
3. $X^\mu = \lambda x^\mu$.
4. $X^\mu = b^\mu x^2 - (b \cdot x)x^\mu$, b is a constant vector.

Notice that 1 and 2 are actual isometries. Now, it's an exercise to look at which algebra these solutions satisfy. We find then that they satisfy the $\mathfrak{so}(p+1, q+1)$ algebra. So, for $\mathbb{R}^{p,q}, p+q > 2$ we have that the isometry group is $SO(p,q)$ and the conformal group is $SO(p+1, q+1)$.

2 Two dimensional case.

Now consider the case $d = 2$ In this case we don't need the metric to be flat. We also assume that our surface is a Riemannian surface (so in particular it is orientable and admits complex coordinates)[1]. Introducing complex coordinates $(z, \bar{z})$ and using the so-called Hermitian coordinates in which the diagonal elements of the metric vanish — $g_{zz} = g_{\bar{z}\bar{z}} = 0$ — we obtain from equation 1.1:

$$\nabla_z X_z = 0$$

Rainsing the index, using that contractions and covariant derivatives commute and also that the diagonal terms of the metric vanish, we obtain that $\nabla_z X^{\bar{z}} = 0$. Since the Christoffel symbol $\Gamma^{\bar{z}}_{z\rho}$ vanishes (direct calculation!), the above equation gives $\partial_z X_{\bar{z}} = 0$, i.e., $X^{\bar{z}}$ is anti-holomorphic. Analogoulsy we obtain that X^z is holomorphic, so the conformal Killing vector field is a sum of a holomorphic and an anti-holomorphic part:

$$X = X^z \frac{\partial}{\partial z} + X^{\bar{z}} \frac{\partial}{\partial \bar{z}}$$

Since the (one-partameter family of) diffeomorphisms generated by X are conformal transformations, we obtain that the conformal transformations are holomorphic and a anti-holomorphic maps, as expected.

[1]In physics non-orientable surfaces also arise. One way of dealing with these is to take the double orientable cover and carry out all the calculations there. In the end we mod out by the appropriate equivalence relation which takes the cover back to the original non-orientable surface.

3 Witt algebra.

Now we want to find the generators; locally we can wirte the coefficients of X as Laurent expansions:

$$X(z) = \sum_{n=-\infty}^{\infty} a_{n+1} z^{n+1}$$
$$X(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{a}_{n+1} \bar{z}^{n+1}$$

where we use $n+1$ instead of n for convenience. So

$$X = \sum_{n=-\infty}^{\infty} \left(a_{n+1} z^{n+1} \frac{\partial}{\partial z} + \bar{a}_{n+1} \bar{z}^{n+1} \frac{\partial}{\partial \bar{z}} \right)$$

Defining $-\ell_n = a_{n+1} z^{n+1} \frac{\partial}{\partial z}$ and $-\bar{\ell}_n = \bar{a}_{n+1} \bar{z}^{n+1} \frac{\partial}{\partial \bar{z}}$, an easy calculation shows

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}$$
$$[\ell_n, \bar{\ell}_m] = 0$$
$$[\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m}$$

the above relations define what is called the *Witt algebra*.

4 Central extensions and Virasoro algebra.

Under the process of quantization, the algebra of classical observers gives rise to an algebra of quantum operators which in general do not commute, and therefore the order in which they appear matters. For example, classically equivalent expressions such as $a_n \bar{a}_m$ and $\bar{a}_m a_n$ give rise, after quantization, to $\hat{a}_n \hat{a}_m^\dagger$ and $\hat{a}_m^\dagger \hat{a}_n$, which are different unless the operators commute[2]. In order to deal with this problem in the quantization process, physicists adopt the following procudure: write the operators always in the same order and then add an extra term to take this ambiguity into account. Matematically this is made by a central extension of the algebra.

[2]physicists use $\hat{a}$ to denote the quantum observable (=operator on a Hilbert space) corresponding to the classical observable (=function) a; they also use † to denote the adjoint operator

Here the classical algebra is the Witt algebra. The corresponding quantum algebra is obtainded by introducing a *central element* I which commutes with every generator (we write L instead of ℓ to stress that now we are dealing with quantum objects).

$$[L_n, I] = 0$$
$$[L_n, L_m] = (n-m)L_{n+m} + C_{nm}I$$

and analogous equations for $\bar{L}_n$. The element I is known as *central charge* or *anomaly*. This new algebra is known as *Virasoro algebra*. Using standard rules for the commutators we can extract some properties of the coefficients C_{mn}:

$$[L_m, L_n] = -[L_n, L_m] \Rightarrow C_{mn} = -C_{nm}$$

Jacobi identity $\Rightarrow (n-m)C_{n+m,k} + (m-k)C_{m+k,n} + (k-n)C_{k+n,m} = 0 \ (*)$

Notice that we can redefine $L'_n = L_n - C_nI$ and get (plugging $L_n = L'_n + C_nI$ into $[L_n, L_m]$ and using $[L_n, I] = 0$):

$$[L'_n, L'_m] = (n-m)L_{n+m} + (n-m)C_{n+m}I$$

So if the coefficients satisfy

$$C_{mn} = (n-m)C_{n+m} \ (**)$$

we get

$$[L'_n, L'_m] = (n-m)L'_{n+m}$$
$$[L'_n, I] = 0$$

So if the coefficients satisfy $(**)$ then the extension is *trivial* in the sense that after redefining the operators the algebra generated by the L's and the algebra generated by I are "decoupled". More precisely, we can think that in this case the extension is the most trivial one: a direct sum of the algebra generated by the L' s with the algebra generated by I. Since we want non-trivial extensions we look for those C_{mn} which are not of the form $(**)$. Define:

$$(\delta C)_{mn} = (m-n)C_{n+m}$$
$$(\delta C)_{mnk} = (m-n)C_{m+n,k} + (n-k)C_{n+k,m} + (k-m)C_{k+m,n}$$

Then a calculation shows that $\delta^2 = 0$. Then we can rewrite $(*)$ and $(**)$ respectively as:

$$(\delta C)_{nmk} = 0 \ (i) \text{ (closedness condition)}$$
$$(\delta C)_{mn} = C_{mn} \ (ii) \text{ (exactness condition)}$$

Therefore, in order to find the non-trivial extensions, we need to find those which satisfies (i) but *not* (ii). In another other, we need to compute the *cohomolgy* of the complex C, i.e., the closed but not exact C's.

Theorem 1. *The only non-trivial extensions are of the form* $C_{mn} = am^3\delta_{m+n,0}$, *a some constant.*

So we have

$$[L_n, L_m] = (n - m)L_{n+m} + a\delta_{m+n,0}m^3 I$$

It is customary to redefine the operators at zero: $L_0 \to L_0 - \frac{b}{2}I$ for some constant b and then put $b = -a$, so we get

$$[L_n, L_m] = (n - m)L_{n+m} + a\delta_{m+n,0}(m^3 - m)I$$

With these redefinitions the central charge for L_{-1}, L_0, L_1 vanishes and these elements satisfies a $\mathfrak{sl}(2)$ algebra (which is useful when we need to find representations).

5 Operator product expansion (OPE).

Let us write the expression for the Virasoro algebra as

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{m,-n}\frac{C}{12}(m^3 - m)I \quad (\star)$$

The constant C is called *central charge*. The idea is that different conformal symmetries give rise to different conformal charges. The factor $\frac{1}{12}$ is introduced for convenience.

Consider a cilindrical worldsheet as shown in figure 1. It can be mapped bi-holomorphically onto an annulus by $z = e^{t+i\sigma}$. Notice that slices $t =$ *constant* are mapped to the circles of radius e^t. The time ordering in the worldsheet becomes radial ordering in the annulus.

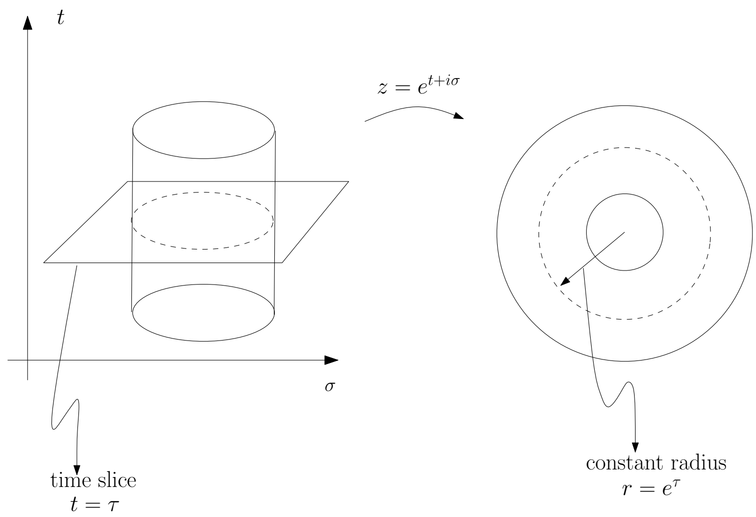

Figure 1: Mapping the worldsheet bi-holomorphically onto the annulus.

Define

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n$$

(from now on analogous formulas will be understood for conjugate quantities). Notice that then

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

In specific models, T is the energy-momentum tensor (more precisely, the T_{zz} component of it). If we want to pursue further this point, it is worth noticing that under the correspondence of circles on the annulus and time slices on the worldsheet, the above integral corresponds to an integral on a time slice, so that L_n can be thought of a conserved charge (since these are time slice integrals of components of the energy-momentum tensor).

For operators A and B which depend on time define their *time ordering* as

$$T(A(t_1)B(t_2)) = \begin{cases} A(t_1)B(t_2) & \text{if } t_1 > t_2 \\ B(t_2)A(t_1) & \text{if } t_1 < t_2 \end{cases}$$

It follows that

$$[A(t), B(t)] = (\lim_{t_1 \searrow t_2} - \lim_{t_1 \nearrow t_2}) T(A(t_1)B(t_2))$$

(notice that the above expression is defined even though for $t_1 = t_2$ T is not). So commutators can be thought of as time-ordering processes. Replacing time-ordering by radial ordering, we can use an analogous expression for defining the *radial ordering* $R(T(z)T(w))$. It follows that for $|z| > |w|$ we have

$$R(T(z)T(w)) = \frac{C/2}{(z-w)^4} I + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{holomorphic part}$$

The expansion above is called the *operator product expansion* and it appears naturally in specific models. Physicists sometimes write the above expression simply as $T(z)T(w) = \dots$, i.e., they omit R.

Theorem 2. *The above operator product expansion is equivalent to* $\star$.

Proof. We are going to show only one implication, i.e., that the operator product expansion above implies the Virasoro algebra $\star$. The other implication is done below after introducing the axioms for a meromorphic conformal field theory.

Write

$$[L_n, L_m] = [\oint \frac{dz}{2\pi i} z^{n+1} T(z), \oint \frac{dw}{2\pi i} w^{m+1} T(w)]$$

where the integration is carried over a circle around zero. Now rewrite the above expression as

$$\oint_{\text{small circle around } 0} \frac{dw}{2\pi i} \Big(\oint_{|z|>|w|} - \oint_{|z|<|w|} \Big) \frac{dz}{2\pi i} z^{n+1} w^{m+1} R(T(z)T(w))$$

Deforming the countour of z-integration as shown in the figure 2 we obtain

$$\oint_{\text{around } 0} \frac{dw}{2\pi i} \oint_{\text{around } w} \frac{dz}{2\pi i} z^{n+1} w^{m+1} R(T(z)T(w)) =$$
$$\oint_{\text{around } 0} \frac{dw}{2\pi i} \oint_{\text{around } w} \frac{dz}{2\pi i} z^{n+1} w^{m+1} \Big(\frac{C/2}{(z-w)^4} I + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \Big)$$

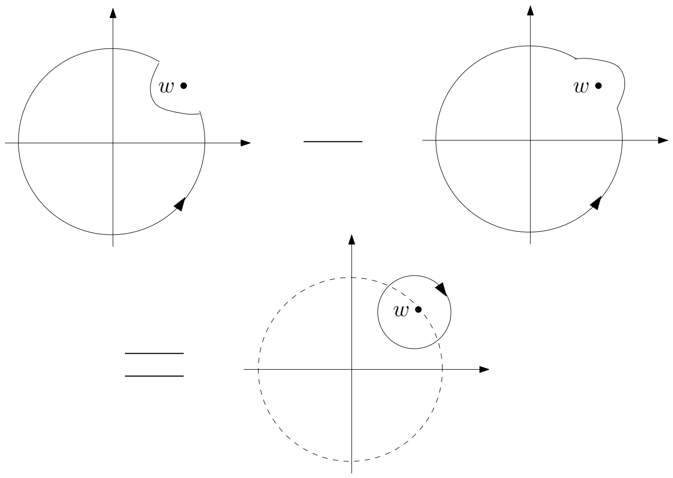

Figure 2: The integral on the path on the left minus the integral on the path on the right equals the integral on the path below.

where we do not write the integral of the holomorphic terms since it is zero because we are integrating on a closed path.

Now expand

$$z^{n+1} = w^{n+1} + (n+1)w^n(z-w) + \frac{1}{2!}(n+1)nw^{n-1}(z-w)^2 + \frac{1}{3!}(n+1)n(n-1)w^{n-2}(z-w)^3 + \dots$$

Then in the z-integral only the residues survive and we get

$$\oint_{\text{around } 0} \frac{dw}{2\pi i}\Big(\frac{C}{12}(n^3-n)w^{n-2} + 2(n+1)T(w)w^n + \partial_w T(w)w^{n+2}\Big)$$

Now recall

$$T(w)\sum_{k=-\infty}^{\infty} w^{-k-2}L_k \Rightarrow \partial_w T(w)\sum_{k=-\infty}^{\infty}(-k-2)w^{-k-3}L_k$$

and plug the above expressions on the integral. Again, only residue terms survive. For example, the first term on the integral gives a residue only when $m+1+n-2=-1$, i.e, $m+n=0$, which gives a term proportional to $\delta_{m,-n}$. Doing something similar for the other terms we get

$$\oint \frac{dw}{2\pi i}(\cdots) = (n-m)L_{n+m} + \delta_{m,-n}\frac{C}{12}(m^3-m)I$$

as desired. □

5.1 Meromorphic Conformal Field Theory (MCFT)

A MCFT is a 5-tuple $(\mathcal{F}, \mathcal{H}, V, |0\rangle, |L\rangle)$ where $\mathcal{H}$ is a Hilbert space, $\mathcal{F}$ is a subspace of $\mathcal{H}$ (thought of as the Fock space or space of states), V is a map $V : \mathcal{F} \times \mathbb{C} \to \text{End}(\mathcal{H})$ (not necessarily linear in the second variable, here $\text{End}(\mathcal{H})$ denotes linear operator densely defined on $\mathcal{H}$), $|0\rangle \in \mathcal{F}$ and $|L\rangle \in \mathcal{F}$ are two distinguished elements (the vacuum and ??), such that the Virasoro algebra is contained in $\text{End}(\mathcal{H})$ and satisfying the following set of axioms:

1. $V(|\phi\rangle, z)|0\rangle = e^{zL_{-1}}|\phi\rangle$.
2. $\langle\phi| V(|\xi\rangle, z)|\psi\rangle$ is a meromorphic function of z.
3. $\langle\phi| V(|\xi\rangle, z)V(|\eta\rangle, z')|\psi\rangle$ is holomorphic for $|z| > |z'|$.
4. $\langle\phi| V(|\xi\rangle, z)V(|\eta\rangle, z')|\psi\rangle = \sum \epsilon \langle\phi| V(|\eta\rangle, z)V(|\xi\rangle, z')|\psi\rangle$, where $\epsilon = 1$ unless both states are fermionic, in which case $\epsilon = -1$.
5. $V(|L\rangle, z) = T(z)$.
6. $L_m|0\rangle = 0$ for $m \geq 0$.

Axiom 1 means that L_{-1} is an evolution operator; 2 is a standard regularity assumption; 3 describes a scattering process, the condition $|z| > |z'|$ meaning causality (recall the correspondence between time ordering and radial ordering); 4 says that the system obeys the standard statistical relations for bosonic and fermionic variables, 5 ?? and 6 means that in the Virasoro algebra L_m, $m \geq 0$ are anihilation operates and L_m, $m < 0$ are creation operators.

We can now prove the other implication of theorem 2, i.e., assuming the Virasoro algebra (and the axioms of MCFT) let us recover the OPE as presented above.

Because $L_{-1}e^{zL_{-1}}|\phi\rangle = \frac{\partial}{\partial z}(e^{zL_{-1}}|\phi\rangle)$ we have $L_{-1} = \frac{\partial}{\partial z}$. Denote by h and $|\psi\rangle$ eigenvalue and eigenvector of L_0, then

$$V(|\psi\rangle, z) = \psi(z) = \sum_{m=-\infty}^{\infty} \psi_m z^{-m-h}$$

$$V(|L\rangle, z) = T(z) = \sum_{m=-\infty}^{\infty} L_m z^{-m-2}$$

We also have $V(|L\rangle, z)|_{z=0} = e^{zL_{-1}}|0\rangle|_{z=0}$, giving $T(z)|0\rangle|_{z=0} = L_{-2}|0\rangle$. Then we have the following OPE:

$$R(T(z)T(z')) = \sum (z-z')^{-m} V(L_{-2+m}|L\rangle, z') = \\ \sum (z-z')^{-m} V(L_{-2+m}L_{-2}|0\rangle, z')$$

Because we are interested only in the poles, we need to figure out only the terms for $m = 1, 2, 3. \dots$. Using the commutation relation of the Virasoro algebra:

$$L_{-2-m}L_{-2} = L_{-2}L_{-2-m} + mL_{m-4} + \delta_{m,4}\frac{C}{2}I$$

Acting with the above expression on the vacuum we get that it equals:

$$L_{-1}L_{-2}|0\rangle \text{ for } m = 1$$
$$(L_{-2}L_0 + 2L_{-2})|0\rangle \text{ for } m = 2$$
$$3L_3|0\rangle \text{ for } m = 3$$
$$(\frac{C}{2}I + 4L_0)|0\rangle \text{ for } m = 4$$
$$0 \text{ for } m = 5, 6, \dots$$

Using the above expressions and $L_{-1}|0\rangle$ we get the desired OPE.

6 Kac-Moody algebra

6.1 Cohomology of Lie algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. By Lie's theorem, it corresponds to a simply connected Lie group G. Since our main object of interest is cohomology with values in $\mathbb{R}$, we define it to be a trivial $\mathfrak{g}$-module, i.e. $\mathfrak{g}$ acts by zero. We will also abbreviate notation and denote $C^k(\mathfrak{g};\mathbb{R})$ by $C^k(\mathfrak{g})$ and the corresponding cohomology groups $H^k(\mathfrak{g};\mathbb{R})$ by $H^k(\mathfrak{g})$.

Remark This works with $\mathbb{C}$-coefficients as well.

The coboundary map is defined by

$$(\delta\omega)(x_0, \dots, x_k) \quad \textstyle\sum_{0\le i<j\le k}(-1)^{i+j}\omega([x_i, x_j], \dots, \hat{x_i}, \dots, \hat{x_j}, \dots). \tag{6.1}$$

Observe that the cohomology groups so obtained are just the the cohomology group of left invariant forms on G and δ is exactly d. By definition, $C^0(\mathfrak{g}) = \mathbb{R}$ and $C^1(\mathfrak{g}) = \mathfrak{g}^* \cong \mathfrak{g}$. The first few coboundary maps translate into

$$(\delta\alpha)(x) = 0, \tag{6.2}$$
$$(\delta\beta)(x,y) = -\beta([x,y]), \tag{6.3}$$
$$(\delta\gamma)(x,y,z) = -\gamma([x,y],z) - \gamma([y,z],x) - \gamma([z,x],y). \tag{6.4}$$

Then (6.2) implies that

$$H^0(\mathfrak{g}) = \mathbb{R}. \tag{6.5}$$

Using (6.3) we see that $H^1(\mathfrak{g})$ is exactly the kernel of $\delta : C^1(\mathfrak{g}) \to C^2(\mathfrak{g})$ since the map $\delta : C^0(\mathfrak{g}) \to C^1(\mathfrak{g})$ is zero. Elements α in the kernel are precisely the ones that vanish on commutators, i.e., $\alpha([x,y]) = 0$ for any $x, y \in \mathfrak{g}$. Alternatively, these can be viewed as maps from $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ to $\mathbb{R}$, whence

$$H^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]. \tag{6.6}$$

In particular, the first cohomology vanishes for a semisimple Lie algebra.

To interpret $H^2(\mathfrak{g})$ we need to understand the kernel of (6.4), i.e., 2-cochains ω such that

$$\omega(([x,y],z) + \omega([y,z],x) + \omega([z,x],y) = 0. \tag{6.7}$$

The restraint above is called the *cocycle condition* and is equivalent to ω being closed. Any such ω defines a central extension

$$0 \to \mathbb{R} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{6.8}$$

with the Lie bracket on $\tilde{\mathfrak{g}}$ given by

$$[(x,s),(y,t)] := ([x,y], \omega(x,y)). \tag{6.9}$$

The bracket satisfies the Jacobi identity due to (6.7) and is skew since ω is. Conversely, given a central extension (6.8), the bracket on $\tilde{\mathfrak{g}}$ is defined as in (6.9) and ω must satisfy (6.7). Thus, the central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are in bijective correspondence with the 2-cocycles. It can be shown that the 2-cocycles ω, ω' are cohomologous if and only if the corresponding central extensions $\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}'$ are equivalent. Thus,

Proposition 6.1.1. *Equivalence classes of central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are in bijective correspondence with elements of $H^2(\mathfrak{g})$.*

If $\mathfrak{g}$ is semisimple, then it turns out that there are no non-trivial central extensions since $H^2(\mathfrak{g}) \cong H^2(G) = 0$.

To discuss $H^3(\mathfrak{g};\mathbb{R})$, we shall restrict ourselves to algebras such that $H^1(\mathfrak{g};\mathbb{R}) = 0 = H^2(\mathfrak{g};\mathbb{R})$. The Lie algebras of any connected compact semisimple Lie group G satisfies this property. Since $\mathfrak{g} \cong \mathfrak{g}^*$ as $\mathfrak{g}$-modules, the space of (symmetric) invariant bilinear forms on $\mathfrak{g}$, $\mathrm{Bil}(\mathfrak{g}) = (S^2\mathfrak{g})^{\mathfrak{g}}$, is isomorphic to $(S^2\mathfrak{g}^*)^{\mathfrak{g}}$. With this identification, define a map

$$\begin{aligned} \varphi : (S^2\mathfrak{g}^*)^{\mathfrak{g}} &\to (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}} \\ B \mapsto \varphi(B) : (x,y,z) &\to B([x,y],z). \end{aligned} \tag{6.10}$$

The 3-form $\varphi(B)$ is antisymmetric since B is invariant and symmetric and $[,]$ is skew. The invariance follows from the Jacobi identity and the invariance of B. The following proposition provides the connection between $\mathrm{Bil}(\mathfrak{g})$ and $H^3(G;\mathbb{R}) \cong (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$.

Proposition 6.1.2. *For a semisimple Lie algebra $\mathfrak{g}$, $\varphi : (S^2\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$ is an isomorphism.*

In view of this result and the discussion preceding it, we conclude that $\mathrm{Bil}(\mathfrak{g})$ is isomorphic to $H^3(G;\mathbb{R})$. If G is simple, then it is one-dimensional since any such bilinear form is a multiple of the Killing form on $\mathfrak{g}$.

6.2 Generalized Cartan matrices

For any finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, let $(,)$ denote the (non-degenerate) Killing form. This is non-degenerate when restricted to $\mathfrak{h}$ and induces a non-degenerate form on $\mathfrak{h}^*$ as well. Fix a reduced root system $R \subset \mathfrak{h}^*$ with a polarization $R = R_- \sqcup R_+$ and the corresponding system of simple roots $\{\alpha_i\}_{i=1}^r$. If one chooses for each root α a non-zero element e_α in the root space $\mathfrak{g}_\alpha$, then $\mathfrak{g}$ is generated by $3r$ generators $\{e_i := e_{\alpha_i}, f_i := e_{-\alpha_i}, h_i := h_{\alpha_i} | \, i = 1,\ldots,r\}$. The subspaces $\mathfrak{n}_\pm$ consisting of positive and negative roots respectively are subalgebras of $\mathfrak{g}$. In fact, $\mathfrak{n}_+$ is generated by the e_i's, $\mathfrak{n}_-$ is generated by the f_i's and h_i's form a basis of $\mathfrak{h}$. In other words

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-. \tag{6.11}$$

After suitably normalizing, the generators satisfy the Jacobi identity and the *Serre relations* (sometimes also known as *Chevalley-Serre relations*)

$$[h_i, h_j] = 0 \tag{6.12}$$
$$[h_i, e_j] = a_{ij} e_j \tag{6.13}$$
$$[h_i, f_j] = -a_{ij} f_j \tag{6.14}$$
$$[e_i, f_j] = \delta_{ij} h_i \tag{6.15}$$
$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0 \tag{6.16}$$
$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \tag{6.17}$$

where the entries of the Cartan matrix $A = (a_{ij})$ are given by

$$a_{ij} = \langle h_{\alpha_i}, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i). \tag{6.18}$$

It is an integer valued $r \times r$ (rank $\mathfrak{g} = r$) matrix satisfying

$$a_{ii} = 2 \tag{6.19}$$
$$a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \tag{6.20}$$
$$a_{ij} \leq 0 \text{ for } i \neq j \tag{6.21}$$
$$\det A > 0. \tag{6.22}$$

This implies that rank $A = r$. Observe that $A = DS$ where

$$D_{ij} = \delta_{ij}/(\alpha_i, \alpha_i), \;\; S_{ij} = 2(\alpha_i, \alpha_j)$$

and S is positive definite since $\{\alpha_i\}$'s span $\mathfrak{h}^*$. Hence the last condition can be replaced by

$$\det A_{\{i\}} > 0, i = 1, \ldots, r \tag{6.23}$$

where $\det A_{\{i\}}$ are the principal minors of A. Since the off-diagonal entries are non-positive integers, it can be shown that

$$a_{ij} \in \{-3, -2, -1, 0, 2\} \tag{6.24}$$

which is equivalent to $\det A > 0$. The *Weyl group* can be constructed directly from the Cartan matrix, viz., the rows of A determine the reflection against the simple roots.

Conversely, given a reduced root system R with a polarization and a set of simple roots $\{\alpha_i\}_{i=1}^r$, one may use the Serre relations to define a Lie algebra $\mathfrak{g}(R)$. It is known that this is isomorphic to a semisimple Lie algebra $\mathfrak{g}$ with root system R, the key step of the proof is showing that $\mathfrak{g}(R)$ is finite dimensional by using the Weyl group. This sets up an isomorphism between the set of isomorphism classes of reduced root systems and the set of isomorphism classes of semisimple Lie algebras.

One may drop the condition (6.22), to define

Definition 6.2.1. A *generalized Cartan matrix* A is an $r \times r$ integer valued matrix satisfying (6.19),(6.20) and (6.21). It is of *finite type* if it satisfies (6.22), i.e., positive definite. It is of *affine type* if all the proper principal minors are positive but $\det A = 0$, i.e., positive semidefinite. Otherwise, it is declared to be of *indefinite/hyperbolic type.*

Given A, the Serre relations define the corresponding abstract complex Lie algebra $\mathfrak{g}(A)$, called a *Kac-Moody algebra.* If A is of finite type, then $\mathfrak{g}(A)$ is finite dimensional semisimple Lie algebra. Otherwise, it is infinite dimensional. The *affine Lie algebras* are the Kac-Moody algebras corresponding to generalized Cartan matrices of affine type. It forms an important subclass of Kac-Moody algebras. If $\mathfrak{g}(A)$ is affine, then A has corank 1, i.e., it's kernel is one dimensional. Define the *Coxeter labels* $(a_i)_{i=1}^r$ and *dual Coxeter labels* $(a^i)_{i=1}^r$ to be the left and right eigenvectors of A respectively with eigenvalue 0, i.e.,

$$\sum_{i=1}^{r} a_i a_{ij} = 0 = \sum_{i=1}^{r} a_{ij} a^j \tag{6.25}$$

together with the normalization condition

$$\min\{a_i | \, i = 1, \ldots, r\} = 1 = \min\{a^i | \, i = 1, \ldots, r\}. \tag{6.26}$$

This makes sense since one can choose the eigenvector in either case to be positive, i.e., $a_i > 0$ (resp. $a^i > 0$).

Let A be indecomposable. If it is of finite type, then $\mathfrak{g}(A)$ is simple, In contrast, if A is of affine type, $\mathfrak{g}(A)$ possess a non-trivial center. For any constant ζ the element

$$k := \zeta \sum_{i=1}^{r} a_i h_i \tag{6.27}$$

is a *central element* since $[k, h_i] = 0$ by definition and

$$[k, e_j] = \zeta \sum_{i=1}^{r} a_i a_{ij} e_j = 0 = -\zeta \sum_{i=1}^{r} a_i a_{ij} f_j = [k, f_j]$$

by (6.25). Since the kernel of A is one dimensional, k spans $Z(\mathfrak{g}(A))$. The existence of such a central element allows for a *central extension* of Lie algebras

$$0 \to \mathbb{C} \overset{k}{\hookrightarrow} \mathfrak{g}(A) \to \mathfrak{g}(A)/\mathbb{C} \to 0. \tag{6.28}$$

We will restrict ourselves to a certain subclass of affine algebras where the role of $\mathfrak{g}(A)/\mathbb{C}$ is played by the *loop algebra* $L\mathfrak{g}$. It is just the Lie algebras associated with the *loop group* LG, the space of smooth maps from S^1 to a Lie group G. We shall further restrict to semisimple groups.

6.3 Affine algebras

At the level of Lie algebras, the central extensions

$$\mathbb{R} \to \widetilde{L\mathfrak{g}} \to L\mathfrak{g}$$

correspond precisely to invariant symmetric bilinear forms on $\mathfrak{g}$. As a vector space $\widetilde{L\mathfrak{g}}$ is $L\mathfrak{g} \oplus \mathbb{R}$, and the bracket is given by

$$[(\xi, \lambda), (\eta, \mu)] = ([\xi, \eta], \omega(\lambda, \mu)) \tag{6.29}$$

for $\xi, \eta \in L\mathfrak{g}$ and $\lambda, \mu \in \mathbb{R}$. Here $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ is the bilinear map

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \, d\theta \tag{6.30}$$

for a chosen symmetric invariant from on $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is semisimple then every invariant bilinear form is symmetric.

For (6.29) to define a Lie bracket, ω must be skew, which follows from integrating by parts in (6.30), and must satisfy the Jacobi/cocycle condition

$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0. \tag{6.31}$$

This can be proved using the fact that $\langle , \rangle$ is invariant and

$$[\xi, \eta]' = [\xi', \eta] + [\xi, \eta'] .$$

If $f \in \mathrm{Diff}^+(S^1)$ then it induces a map $f^* : L\mathfrak{g} \to L\mathfrak{g}$ via precomposing. Then

$$\omega(f^*\xi, f^*\eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(f(\theta)), \eta'(f(\theta)) \rangle \, f'(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\tau), \eta'(\tau) \rangle \, d\tau = \omega(\xi, \eta), \tag{6.32}$$

where $\tau = f(\theta)$. Thus, ω is $\mathrm{Diff}^+(S^1)$ invariant and $\mathrm{Diff}^+(S^1)$ acts as a group of diffeomorphisms on $\widetilde{L\mathfrak{g}}$. We will see later that it also acts on the group extension. Notice that ω is invariant under conjugation by constant loops, i.e.,

$$\omega(g\xi, g\eta) = \omega(\xi, \eta)$$

since $\langle , \rangle$ is invariant (here $g\xi$ denotes the adjoint action).

There are essentially no other cocycles other than ω defined by (6.30). More precisely, we may only consider invariant cocycles since if α is a cocycle, then $g\alpha$ defined by $g\alpha(\xi, \eta) = \alpha(g^{-1}\xi, g^{-1}\eta)$ defines an equivalent extension, viz,

$$g \oplus \mathrm{Id} : (L\mathfrak{g} \oplus \mathbb{R}, \alpha) \to (L\mathfrak{g} \oplus \mathbb{R}, g\alpha), \ \ (\xi, \lambda) \mapsto (g\xi, \lambda)$$

is an isomorphism of Lie algebras. Similarly, the extension given by the invariant cocycle

$$\int_G g\alpha \, d\mu$$

is isomorphic to the one given by α. We have

Proposition 6.3.1. *If $\mathfrak{g}$ is semisimple then the only continuous G-invariant cocycles on $L\mathfrak{g}$ are given by* (6.30).

Proof Any cocycle $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ can be extended to a complex bilinear map $\omega : L\mathfrak{g}_C \times L\mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. Since an element $\xi \in L\mathfrak{g}_{\mathbb{C}}$ can be extended in a Fourier series $\sum \xi_k z^k$ with $\xi_k \in \mathfrak{g}_{\mathbb{C}}$, by continuity ω is determined by values on $\xi_k z^k$. Write $\omega_{p,q}(\xi, \eta) = \omega(\xi_p z^p, \eta_q z^q)$ for $\xi, \eta \in \mathfrak{g}_{\mathbb{C}}$; this is a G-invariant bilinear map $\mathfrak{g}_C \times \mathfrak{g}_C \to \mathbb{C}$ which is necessarily symmetric since $\mathfrak{g}_{\mathbb{C}}$ is semisimple. Then

$$\omega_{p,q}(\xi, \eta) = -\omega(\eta z^q, \xi z^p) = -\omega_{q,p}(\eta, \xi) = -\omega_{q,p}(\xi, \eta).$$

The cocycle identity translates to

$$\omega_{p+q,r} + \omega_{q+r,p} + \omega_{r+p,q} = 0, \ \ \forall \, p, q, r. \tag{6.33}$$

It can be shown that $\omega_{p,q} = 0$ if $p + q \neq 0$ and $\omega_{p,-p} = p\,\omega_{1,-1}$. If we write $\xi = \sum \xi_p z^p$ and $\eta = \sum \eta_q z^q$, then

$$\omega(\xi, \eta) = \sum_p p\,\omega_{1,-1}(\xi_p, \eta_{-p}). \tag{6.34}$$

On the other hand

$$\begin{aligned}\frac{i}{2\pi}\int_0^{2\pi}\omega_{1,-1}(\xi(\theta),\eta'(\theta))d\theta &= \sum_{p,q}\frac{-1}{2\pi}\int_0^{2\pi} q\omega_{1,-1}(\xi_p,\eta_q)e^{i(p+q)\theta}d\theta \\ &= \sum_p \frac{p}{2\pi}\int_0^{2\pi}\omega_{1,-1}(\xi_p,\eta_{-p})d\theta,\end{aligned}$$

which equals (6.34). Thus, ω is completely determined by $\omega_{1,-1}$ and is of the form (6.30). □

Notation Let $\mathrm{Bil}(\mathfrak{g})$ denote the space of invariant bilinear forms on $\mathfrak{g}$ and K its dual. We also denote the central extension $\widetilde{L\mathfrak{g}}$ of $L\mathfrak{g}$ corresponding to any $B \in \mathrm{Bil}(\mathfrak{g})$ by $\tilde{\mathfrak{g}}_B$.

The above result determines the universal central extension of $L\mathfrak{g}$, viz,

$$0 \to K \to \tilde{\mathfrak{g}}_{\mathrm{univ}} \to L\mathfrak{g} \to 0 \tag{6.35}$$

and it is universal because any extension of $L\mathfrak{g}$ by $\mathbb{R}$ (corresponding to $B \in \mathrm{Bil}(\mathfrak{g})$) can be obtained as the push-forward

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \tilde{\mathfrak{g}}_{\mathrm{univ}} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle B} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{\mathfrak{g}}_B & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \end{array} \tag{6.36}$$

Now assume that $\mathfrak{g}$ is a semisimple Lie algebra corresponding to a connected compact Lie group G. Then it follows from §1.5 (6.1.2) that $K \cong H^3(G;\mathbb{R})$. If G is simple then $H^3(G;\mathbb{R}) = \mathbb{R}$, whence only one central extension exist up to scaling.

References

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