COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **18**, Number **4**, July **2019**

doi:10.3934/cpaa.2019075

pp. **1567–1599**

ON THE EXISTENCE OF SOLUTIONS AND CAUSALITY FOR RELATIVISTIC VISCOUS CONFORMAL FLUIDS

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(Communicated by Igor Kukavica)

Abstract. We consider a stress-energy tensor describing a pure radiation viscous fluid with conformal symmetry introduced in [3]. We show that the corresponding equations of motions are causal in Minkowski background and also when coupled to Einstein's equations, and solve the associated initial-value problem.

1. **Introduction.** Consider the following stress-energy tensor for a relativistic fluid with viscosity:

$$\begin{aligned} T_{\alpha\beta} =& \frac{4}{3}u_\alpha u_\beta \epsilon + \frac{1}{3}g_{\alpha\beta}\epsilon - \eta\pi_\alpha^\mu \pi_\beta^\nu(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda) \\ &+ \lambda(u_\alpha u^\mu \nabla_\mu u_\beta + u_\beta u^\mu \nabla_\mu u_\alpha) + \frac{1}{3}\chi\pi_{\alpha\beta}\nabla_\mu u^\mu + \chi u_\alpha u_\beta \nabla_\mu u^\mu \\ &+ \frac{\lambda}{4\epsilon}(u_\alpha \pi_\beta^\mu \nabla_\mu \epsilon + u_\beta \pi_\alpha^\mu \nabla_\mu \epsilon) + \frac{3\chi}{4\epsilon}u_\alpha u_\beta u^\mu \nabla_\mu \epsilon + \frac{\chi}{4\epsilon}\pi_{\alpha\beta}u^\mu\nabla_\mu\epsilon. \end{aligned} \tag{1}$$

Here, u is the four-velocity of fluid particles, normalized so that

$$u^\alpha u_\alpha = -1, \tag{2}$$

ϵ is the energy density of the fluid, g is a (Lorentzian) metric, ∇ is the Levi-Civita connection associated with g, $\pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, and η, λ, and χ are viscous transport coefficients — so that $\eta = \lambda = \chi = 0$ corresponds to an ideal fluid. The transport coefficients are non-negative functions of ϵ. Coefficient η is the usual coefficient of shear viscosity, whereas λ and χ are related to relaxation times. More precisely, while λ and χ, differently than η, have no analogue in more familiar theories such as classical, non-relativistic Navier-Stokes, their physical meaning can be understood from the derivation of (1) from kinetic theory given in [3]. In that case, one may interpret $\lambda/(s\theta)$ and $\chi/(s\theta)$, where s is the entropy density and θ the temperature, as relaxation times that restore causality (since intuitively causality says that the system needs some time to relax back to equilibrium after a perturbation). See [3] for details.

2000 *Mathematics Subject Classification.* Primary: 35Q76; Secondary: 35L15.

Key words and phrases. Relativistic viscous fluids, conformal fluids, Einstein's equations, causality, well-posedness.

M. M. D. is partially supported by NSF grant # DMS-1812826, by a Sloan Research Fellowship provided by the Alfred P. Sloan foundation, and by a Discovery grant administered by Vanderbilt University.

We are interested in the case of pure radiation, when the fluid's pressure is given by $p = \frac{1}{3}\epsilon$, and, therefore, p has already been eliminated from $T_{\alpha\beta}$.

Above and throughout, we adopt the following:

Convention 1. We work in units where $8\pi G = c = 1$, where G is Newton's constant and c is the speed of light in vacuum. Our signature for the metric is $-+++$. Greek indices run from 0 to 3 and Latin indices from 1 to 3.

We shall couple (1) to Einstein's equations:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta}, \tag{3}$$

where $R_{\alpha\beta}$ and R are, respectively, the Ricci and scalar curvature of the metric g, and Λ is a constant (the cosmological constant). We recall that in light of the Bianchi identities, a necessary condition for (3) to hold is that

$$\nabla_\alpha T^\alpha_\beta = 0. \tag{4}$$

Naturally, equations (3)-(4) are defined in a four-dimensional differentiable manifold, the space-time.

We shall establish the following.

Main result. *(see Theorems 2.2 and 2.3 for precise statements) Under appropriate conditions on the initial data and the transport coefficients, the system of Einstein's equations coupled to (1) is causal and admits a unique solution. Causality and uniqueness are here understood in the usual sense of general relativity. Existence, uniqueness, and causality remain true if we consider solely (4) in Minkowski space-time.*

The tensor (1) was introduced[1] in [3]. As discussed there, (1) is the first example in the literature of a stress-energy tensor for relativistic viscous fluids satisfying the following list of physical requirements: in Minkowski background, equations (4) are (i) linearly stable with respect to perturbations around homogeneous thermodynamic equilibrium, (ii) well-posed, and (iii) causal; (iv) Einstein's equations coupled to (1) are well-posed and causal; (v) equations (4) reduce to the standard Navier-Stokes equations in the non-relativistic limit; (vi) an out-of-equilibrium entropy can be defined so that solutions to (4) satisfy the (out of equilibrium) second law of thermodynamics; and (vii) $T_{\alpha\beta}$ can be derived from microscopic kinetic theory.

One reason for seeking a stress-energy tensor satisfying the above properties is that the traditional forms of the relativistic Navier-Stokes equations fail to be causal and stable [23, 35], and attempts to construct a relativistic viscous theory satisfying (i)-(vi) have been limited so far[2]. See [12, 15, 16, 37] for a discussion. In [3] it is also shown that $T_{\alpha\beta}$ yields a well-defined temperature in the test-case of the Gubser flow, in contrast to the traditional relativistic Navier-Stokes' equations that yield a negative temperature, and that a hydrodynamic attractor exists for the dynamics of the Bjorken flow.

Tensor (1) describes a conformal fluid. Loosely speaking, this means that (1) is well-behaved under conformal changes of the metric. More precisely, consider

[1]In [3], (1) is written in a different form, using the so-called Weyl derivative (whose definition is given in [3]; see [33] for more details) instead of the covariant derivative. Both expressions agree once the Weyl derivative is expanded in terms of the covariant derivative.

[2]It is interesting to note that the seemingly easier task of generalizing the non-relativistic Navier-Stokes to Riemannian manifolds is not without problems either, see [5].

a conformal transformation $g'_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta}$, and the transformed quantities $u'_\alpha = e^{-\phi} u_\alpha$, $\epsilon' = e^{4\phi}\epsilon$. Then the fluid is called conformal if $T_{\alpha\beta}$ is traceless and the corresponding transformed $T'_{\alpha\beta}$ satisfies

$$T'_{\alpha\beta} = e^{2\phi} T_{\alpha\beta}.$$

One can show [2, 4] that under these conditions

$$\nabla'_\alpha (T')^\alpha_\beta = e^{4\phi} \nabla_\alpha T^\alpha_\beta,$$

so in particular solutions are preserved by the above transformations. There exists a large literature on conformal fluids and their applications in physics, to which the reader is referred for a discussion (see, e.g., [11, 20] and references therein; for the mathematical background for these references, see [19]). We restrict ourselves to mentioning that conformal fluids are of importance in the study of the quark-gluon plasma that forms in high-energy collisions of heavy-ions; the quark-gluon plasma at very high temperatures is the prototypical example of a relativistic viscous fluid with an equation of state of pure radiation.

The definition of conformal fluid, stated above, will play no direct role in this work per se. Rather, we shall use one of its main consequences, namely, that for such fluids we have

$$\chi = a_1 \eta, \lambda = a_2 \eta, \tag{5}$$

where a_1 and a_2 are constants. Therefore all transport coefficients are determined once we are given $\eta = \eta(\epsilon)$.

Our main result has previously appeared in [3], but the letter format of that manuscript and the fact that it was addressed primarily to a physical audience prevented us from presenting several details of the proof. In particular, the argument in [3] may not be entirely satisfactory for a mathematical audience.

Definition 1.1. For the rest of the the paper, we shall refer to the system of equations (3), with $T_{\alpha\beta}$ given by (1) and u satisfying (2), as the viscous Einstein-conformal fluid (VECF) system.

2. Statement of the results.

We now turn to the precise formulation of the Main Result. We begin by discussing the initial data for the VECF system.

Definition 2.1. An initial data set for the VECF system consists of a three-dimensional smooth manifold Σ, a Riemannian metric g_0 on Σ, a symmetric two-tensor κ on Σ, two real-valued functions ϵ_0 and ϵ_1 defined on Σ, and two vector fields v_0 and v_1 on Σ, such that the Einstein constraint equations are satisfied.

We recall that the constraint equations are given by the following system of equations on Σ:

$$R_{g_0} - |\kappa|^2_{g_0} - (\operatorname{tr}_{g_0} \kappa)^2 = 2\rho$$
$$\nabla_{g_0} \operatorname{tr}_{g_0} \kappa - \operatorname{div}_{g_0} \kappa = j$$

where R_{g_0} is the scalar curvature of g_0, ∇_{g_0}, tr_{g_0}, div_{g_0}, and $|\cdot|_{g_0}$ are the covariant derivative, trace, divergence, and norm with respect to g_0. The quantities ρ and j are given by $\rho = T(n,n)$ and $j = T(n,\cdot)$, where n is the future-pointing unit normal to Σ inside a development of the initial data and T is the stress-energy tensor.

Because $T_{\alpha\beta}$ involves first derivatives of u and ϵ, initial conditions for their time derivatives have to be given, hence the necessity of two functions and two vector fields. Even though u is a four-vector, it suffices to specify vector fields on Σ, with initial conditions for the non-tangential components of u derived from (2) (see section 3.2). It is well-known that initial data for Einstein's equations cannot be prescribed arbitrarily, having to satisfy the associated constraint equations, see, e.g., [21], for details.

We can now state our main result. The definition of spaces G^s and $G^{m,s}$ is recalled in Appendix A.1. We refer the reader to the general relativity literature (e.g., [7, 21, 25, 38, 40]) for the terminology employed in Theorem 2.2.

Theorem 2.2. *Let $\mathcal{I} = (\Sigma, g_0, \kappa, \epsilon_0, \epsilon_1, v_0, v_1)$ be an initial data set for the VECF system. Assume that Σ is compact with no boundary, and that $\epsilon_0 > 0$. Suppose that χ and λ are given by (5), where $\eta : (0,\infty) \to (0,\infty)$ is analytic, and assume that $a_1 = 4$ and $a_2 \geq 4$. Finally, assume that the initial data is in $G^{(s)}(\Sigma)$ for some $1 < s < \frac{17}{16}$. Then:*

1) There exists a globally hyperbolic development M of $\mathcal{I}$.

2) M is causal, in the following sense. Let (g, ϵ, u) be a solution to the VECF system provided by the globally hyperbolic development M. For any $p \in M$ in the future of Σ, $(g(p), u(p), \epsilon(p))$ depends only on $\mathcal{I}|_{i(\Sigma)\cap J^-(p)}$, where $J^-(p)$ is the causal past of p and $i : \Sigma \to M$ is the embedding associated with the globally hyperbolic development M.

We note that, in the standard PDE language, Theorem 2.2 is local in time. But as usual in general relativity, solutions to Einstein's equations are geometric (a solution to Einstein's equations is a Lorentzian manifold) and, in particular, coordinate independent, whereas a statement like "there exists a $\mathcal{T} > 0$..." (as is usual in local in time results) requires the introduction of coordinates. This is why the theorem is better stated as the existence of a globally hyperbolic development[3]. We assumed that Σ is compact for simplicity, otherwise asymptotic conditions would have to be prescribed. The type of asymptotic conditions one would impose had Σ been non-compact depends on the type of questions one is investigating. For instance, it is customary to require g_0 to be asymptotically flat, but other conditions, such as asymptotically hyperbolic, are often used. As for the matter variables, several choices are possible. One can require v_0 and ϵ_0 to approach zero, a constant, or some other specified profile at infinity. The literature on Einstein's equations with non-compact Σ is vast, and a discussion of asymptotic conditions can be found, e.g., [7, 8] and references therein. The assumption $\epsilon_0 > 0$ in Theorem 2.2 (which implies a uniform bound from below away from zero by the compactness of Σ), however, is crucial. This is apparent from expression (1), but it is worth mentioning that allowing ϵ_0 to vanish leads to severe technical difficulties even in the better studied case of the Einstein-Euler system (see [18, 24, 36] for the known results and [13] for a discussion; in fact, the difficulties with vanishing density are present already in the non-relativistic case, see the discussion in [14, 31]). In particular, if we were dealing with a non-compact Σ and had chosen an asymptotic condition where ϵ_0 approaches zero, the techniques here employed would not directly apply.

[3] We recall that a globally hyperbolic development is, roughly speaking, a Lorentzian manifold where Einstein's equations are satisfied and in which Σ embeds isometrically as a Cauchy surface taking the correct data. We also recall that once a globally hyperbolic development is shown to exist, one can prove the existence of the "largest" possible global hyperbolic development, i.e., the maximal globally hyperbolic development of the initial data, which is (geometrically) unique. See [25, 38] for details.

The assumptions $a_1 = 4$ and $a_2 \geq 4$ are technical[4], but they are consistent with conditions that guarantee the previously mentioned linear stability of (1). Note that while our proof is restricted to the Gevrey class, our result guarantees that causality will be automatically satisfied in any function space where uniqueness can be established. This is relevant in view of the difficulties of constructing causal theories of relativistic viscous fluids.

Next, we consider the case of a Minkowski background.

Theorem 2.3. *Let T be given by (1) with g being the Minkowski metric. Suppose that χ and λ satisfy (5), with $a_1 = 4$, $a_2 \geq 4$, where $\eta : (0,\infty) \to (0,\infty)$ is a given analytic function. Let $\epsilon_0, \epsilon_1 : \mathbb{R}^3 \to \mathbb{R}$ and $v_0, v_2 : \mathbb{R}^3 \to \mathbb{R}^3$ belong to $G^{(s)}(\mathbb{R}^3)$ for some $1 \leq s < \frac{7}{6}$, and assume that $\epsilon_0 \geq C_0 > 0$, where C_0 is a constant.*

Then, there exists a $\mathcal{T} > 0$, a function $\epsilon : [0,\mathcal{T}) \times \mathbb{R}^3 \to (0,\infty)$, and a vector field $u : [0,\mathcal{T}) \times \mathbb{R}^3 \to \mathbb{R}^4$, such that (ϵ, u) satisfies equations (2) and (4) in $[0,\mathcal{T}) \times \mathbb{R}^3$, $\epsilon(0,\cdot) = \epsilon_0$, $\partial_0\epsilon(0,\cdot) = \epsilon_1$, $u(0,\cdot) = u_0$, and $\partial_0 u(0,\cdot) = u_1$, where ∂_0 is the derivative with respect to the first coordinate in $[0,\mathcal{T}) \times \mathbb{R}^3$. This solution belongs to $G^{2,(s)}([0,\mathcal{T}) \times \mathbb{R}^3)$ and is unique in this class. Finally, the solution is causal, in the following sense. For any $p \in [0,T) \times \mathbb{R}^3$, $(\epsilon(p), u(p))$ depends only on $(\epsilon_0, \epsilon_1, v_0, v_1)|_{\{x^0=0\}\cap J^-(p)}$, where $J^-(p)$ is the causal past of p (with respect to the Minkowski metric).

While formally Theorem 2.3 can not be derived as a corollary of Theorem 2.2, its validity should come as no surprise once we know the latter to be true. In fact, the proof of Theorem 2.3 will be essentially contained in that of Theorem 2.2, as we shall see. It is nonetheless useful to state Theorem 2.3 given the importance of viscous fluids in Minkowski background for applications.

Remark 1. The difference between $s > 1$ in Theorem 2.2 and $s \geq 1$ in Theorem 2.3 comes from the fact that in the proof of Theorem 2.2 we work in local coordinates and employ bump functions, which cannot be analytic (case $s = 1$). In Minkowski space, however, we can use global coordinates and analyticity is not prevented.

3. Proof of Theorem 2.2.

In this section we prove Theorem 2.2, thus we henceforth assume its hypotheses. We will always denote by s a number in $(1, \frac{17}{16})$, as in the statement of the theorem. The proof will be split in several parts. Some of the arguments parallel well-known constructions in general relativity in the smooth setting, but we present them because some additional steps are required in the Gevrey class.

3.1. The equations of motion.

Here we write the VECF in coordinates and in a more explicit form. At this point, we are only interested in writing the equations in a suitable form, thus we assume the validity of (2) and (3) (and consequently (4)), and derive relations of interest.

As is customary, we shall write (3) in trace-reversed form and in wave coordinates. More precisely, we consider the reduced Einstein equations given by

$$g^{\mu\nu}\partial^2_{\mu\nu}g_{\alpha\beta} = B_{\alpha\beta}(\partial\epsilon, \partial u, \partial g), \tag{6}$$

where above and henceforth we adopt the following:

[4] Other values of a_1 and a_2 are in fact possible as showed in [3], and the proof for these other cases is essentially the same as showed here. The main difference is how one factors the characteristic determinant. This different factorization is carried out in [3]. See Remark 16.

Notation 1. We shall employ the letters B and $\widetilde{B}$, with indices attached when appropriate, to denote a general expression depending on at most the number of derivatives indicated in its argument. For instance, in (6), $B_{\alpha\beta}$ represents an expression depending on at most first derivatives of ϵ, first derivatives of u, and first derivatives of g. As another example, $\widetilde{B}(\epsilon, \partial u, \partial^2 g)$ denotes an expression depending on at most zero derivatives of ϵ, one derivative of u, and two derivatives of g. B and $\widetilde{B}$ can vary from expression to expression. It can be easily verified that B and $\widetilde{B}$ will always be an analytic function (typically involving only products and quotients) of its arguments.

Equations (4) become[5]

$$\begin{aligned}(-\eta g^{\alpha\mu} &+ (\lambda - \eta)u^\alpha u^\mu)\partial^2_{\alpha\mu}u^\beta + (\lambda+\chi)u^\beta u^\mu \partial^2_{\mu\alpha}u^\alpha + \frac{1}{3}(-\eta+\chi)g^{\beta\mu}\partial^2_{\mu\alpha}u^\alpha \\ &+ \frac{1}{3}(-\eta+\chi)u^\beta u^\mu \partial^2_{\mu\alpha}u^\alpha + \frac{1}{4\epsilon}u^\beta(\lambda g^{\alpha\mu} + (\lambda+3\chi)u^\alpha u^\mu)\partial^2_{\alpha\mu}\epsilon \\ &+ \frac{1}{4\epsilon}(\lambda+\chi)u^\alpha g^{\beta\mu}\partial^2_{\alpha\mu}\epsilon + \frac{1}{4\epsilon}(\lambda+\chi)u^\beta u^\alpha u^\mu \partial^2_{\alpha\mu}\epsilon + \widetilde{B}^\beta(\partial u, g)\partial^2 g \\ =&B^\beta(\partial\epsilon, \partial u, \partial g).\end{aligned} \tag{7}$$

The term $\widetilde{B}^\beta(\partial u, g)\partial^2 g$, which is linear in $\partial^2 g$, comes from derivatives of the Christoffel symbols, after expanding the second covariant derivatives of u. This term is of the form $\widetilde{B}^\beta(\partial u, g, \partial^2 g)$ according to Notation 1, but we wrote it as $\widetilde{B}^\beta(\partial u, g)\partial^2 g$ to emphasize that we shall consider it as a second order quasi-linear operator on g. The particular form of this operator will not be needed, but it is important that it be included in the principal part of the system for the derivative counting employed below.

Applying $u^\alpha u^\mu \nabla_\alpha \nabla_\mu$ to (2) produces

$$u_\lambda u^\alpha u^\mu \partial^2_{\alpha\mu}u^\lambda + \widetilde{B}(\partial u, g)\partial^2 g = B(\partial u, \partial g). \tag{8}$$

We introduce the vector

$$U = (u^\beta, \epsilon, g_{\alpha\beta}),$$

where we adopt the obvious notation with u^β denoting (u^0, u^1, u^2, u^3), etc.; such a notation is used throughout, including in the matrices below. We write equations (6), (7), and (8) in matrix form as

$$\mathfrak{M}(U, \partial)U = \mathfrak{q}(U), \tag{9}$$

where

$$\mathfrak{M}(U, \partial) = \begin{pmatrix} m(U,\partial) & b(U,\partial) \\ 0 & g^{\mu\nu}\partial^2_{\mu\nu}\end{pmatrix} \tag{10}$$

with

$$\begin{aligned} m_{00}(U,\partial) =&(-\eta g^{\alpha\mu} + (\lambda-\eta)u^\alpha u^\mu)\partial^2_{\alpha\mu} + (\lambda+\chi)u^0 u^\alpha \partial^2_{0\alpha} \\ &+ \frac{1}{3}(-\eta+\chi)(g^{0\alpha} + u^0 u^\alpha)\partial^2_{0\alpha}, \end{aligned}$$

$$m_{0i}(U,\partial) =(\lambda+\chi)u^0 u^\alpha \partial^2_{\alpha i} + \frac{1}{3}(-\eta+\chi)(g^{0\alpha} + u^0 u^\alpha)\partial^2_{\alpha i},$$

[5] See Appendix B for a derivation of (6) and (7).

$$m_{i\nu}(U,\partial) = u^i(\lambda+\chi)u^\alpha\partial^2_{\alpha\nu} + \frac{1}{3}(-\eta+\chi)(g^{i\alpha}+u^iu^\alpha)\partial^2_{\alpha\nu},\ \nu\neq i,$$

$$\begin{aligned} m_{ii}(U,\partial) =& (-\eta g^{\alpha\mu} + (\lambda-\eta)u^\alpha u^\mu)\partial^2_{\alpha\mu} + u^i(\lambda+\chi)u^\alpha\partial^2_{\alpha i} \\ &+ \frac{1}{3}(-\eta+\chi)(g^{i\alpha}+u^iu^\alpha)\partial^2_{\alpha i}, \\ &\text{with no sum over } i, \end{aligned}$$

$$m_{\nu 4}(U,\partial) = \frac{1}{4\epsilon}u^\nu(\lambda g^{\alpha\mu} + (\lambda+3\chi)u^\alpha u^\mu)\partial^2_{\alpha\mu} + \frac{1}{4\epsilon}(\lambda+\chi)(u^\alpha g^{\nu\mu} + u^\nu u^\alpha u^\mu)\partial^2_{\alpha\mu},$$

$$m_{4\nu}(U,\partial) = u_\nu u^\alpha u^\mu \partial^2_{\alpha\mu}.$$

(Recall Convention 1: above we have $1\leq i\leq 3$.) The matrix $b(U,\partial)$ in (10) corresponds to the matrix with the operators $\widetilde{B}^\beta(\partial u, g)\partial^2$ and $\widetilde{B}(\partial u, g)\partial^2$ that act on g (see (7) and (8)), whose explicit form will not be important here. Finally, $g^{\mu\nu}\partial^2_{\mu\nu}$ in (10) represents the 10×10 identity matrix times the operator $g^{\mu\nu}\partial^2_{\mu\nu}$. The vector $\mathfrak{q}(U)$ corresponds to the right-hand side of equations (6), (7), and (8), i.e.,

$$\mathfrak{q}(U) = (B^\beta(\partial\epsilon,\partial u,\partial g), B(\partial u, g), B_{\alpha\beta}(\partial\epsilon,\partial u,\partial g)).$$

3.2. **Initial data.** We now investigate the appropriate initial conditions for (9). We remind the reader that the geometric data in the assumptions of Theorem 2.2 are intrinsic to Σ, thus they do not determine full data for the system[6]. Hence, we need to complete the given data to a full set of initial data.

Assume that $\mathcal{I}$ is given as in the statement of Theorem 2.2. Embed Σ into $\mathbb{R}\times\Sigma$ and consider $p\in\{0\}\times\Sigma$. We shall initially obtain a solution in a neighborhood of p, hence we prescribe initial data locally.

Take coordinates $\{x^\alpha\}_{\alpha=0}^3$ in a neighborhood $\mathcal{U}$ of p such that $\{x^i\}_{i=1}^3$ are coordinates on Σ, which we assume to be normal coordinates for g_0 centered at p. We remark that in these coordinates the initial data will be in $G^{(s)}(\{x^0=0\}\cap\mathcal{U})$. For, by our assumption on $\mathcal{I}$, there exist local coordinates $\{y^i\}_{i=1}^3$ in a neighborhood $\mathcal{Y}\subseteq\Sigma$ of p such that, in these coordinates, the initial data is Gevrey regular. One obtains (short-time) geodesics starting at p by solving the geodesic equation, which will be an ODE with Gevrey data in the $\{y^i\}$ coordinates. Since we can equip Gevrey spaces with a norm, the usual Picard iteration can be applied to solve the geodesic equation, and hence we obtain solutions that are Gevrey regular and vary within the Gevrey class with the initial data. Therefore, the exponential map and, as a consequence, the coordinates $\{x^i\}$ are Gevrey regular in $\mathcal{Y}$ with respect to the $\{y^i\}$ coordinates. Expressing the initial data now in $\{x^i\}$ coordinates, we conclude from standard properties of composition and products of Gevrey maps (see, e.g., [32]) that the initial data is in $G^{(s)}(\{x^0=0\}\cap\mathcal{U})$ in the $\{x^i\}$ coordinates.

We prescribe the following initial conditions for $g_{\alpha\beta}$ on $\{x^0=0\}\cap\mathcal{U}$:

$$g_{ij}(0,\cdot) = (g_0)_{ij},\ g_{00}(0,\cdot) = -1,\ g_{0i}(0,\cdot) = 0,\ \partial_0 g_{ij}(0,\cdot) = \kappa_{ij},$$

[6]For example, g_0 is a metric on Σ which is a three-manifold; thus, g_0 contains only nine (six independent) components locally, whereas there are sixteen (ten independent) components in the full space-time metric. Similarly, κ does not determine all transversal derivatives of g on Σ, and v_0 and v_1 determine only the initial three-velocity and its transversal derivatives, whereas we need the four-velocity u and its tranversal derivatives initially. These mismatches are, as it is well-known, related to the gauge freedom of Einstein's equations. See, e.g., [7] for more discussion.

and $\partial_0 g_{0\alpha}(0,\cdot)$ is chosen such that $\{x^\alpha\}$ are wave coordinates for g at $x^0=0$ (which is well-known to always be possible).

For u^β, we prescribe

$$u^i(0,\cdot)=v_0^i,\ u^0(0,\cdot)=\sqrt{1+(g_0)_{ij}v_0^iv_0^j},\ \partial_0 u^i(0,\cdot)=v_1^i,$$
$$\partial_0 u^0(0,\cdot)=\frac{1}{\sqrt{1+(g_0)_{ij}v_0^iv_0^j}}\Bigg((g_0)_{ij}v_0^jv_1^i+\frac{1}{2}\kappa_{ij}v_0^iv_0^j+\frac{1}{2}\partial_0 g_{00}(0,\cdot)(1+(g_0)_{ij}v_0^iv_0^j)$$
$$+\partial_0 g_{0i}(0,\cdot)v_0^i\sqrt{1+(g_0)_{ij}v_0^iv_0^j}\Bigg).$$

(Note that the radicands are non-negative because g_0 is a Riemannian metric.) The initial conditions for u^0 and $\partial_0 u^0$ have been derived from (2) and the above initial conditions for $g_{\alpha\beta}$. Finally,

$$\epsilon(0,\cdot)=\epsilon_0,\ \partial_0\epsilon(0,\cdot)=\epsilon_1.$$

3.3. **Initial conditions for the system in $\mathbb{R}^4$.** Consider the local coordinates introduced in section 3.2. Via these coordinates and identifying p with the origin, we can regard system (9) as defined in an open set $\mathcal{U}$ of $\mathbb{R}^4$ containing the origin, with the initial conditions prescribed on $\{x^0=0\}\cap\mathcal{U}$. Note that we can also take (9) as a system of equations on the whole of $\mathbb{R}^4$, and we therefore do so. We seek to extend the initial data to the whole hypersurface $\{x^0=0\}$, thus determining initial conditions for the system in $\mathbb{R}^4$.

Let $\mathcal{V}$ be compactly contained in $\{x^0=0\}\cap\mathcal{U}$ and $\mathcal{W}$ be compactly contained in $\mathcal{V}$. Let $\varphi:\{x^0=0\}\to\mathbb{R}$ be a function in $G^{(s)}(\mathbb{R}^3)$ such that $0\leq\varphi\leq 1$, $\varphi=1$ in $\mathcal{W}$, and $\varphi=0$ in the complement of $\mathcal{V}$. Denote by h the Minkowski metric and set, on $\{x^0=0\}$,

$$\mathring{g}_{ij}=\varphi(g_0)_{ij}+(1-\varphi)h_{ij},\ \mathring{g}_{00}=-1,\ \mathring{g}_{0i}=0,\ \partial_0\mathring{g}=\varphi\kappa_{ij}.$$

These will be initial conditions for $g_{\alpha\beta}$ (for equations (9) in $\mathbb{R}^4$), with an usual abuse of notation to denote the initial conditions involving ∂_0. As our coordinates have been chosen with $\{x^i\}$ normal coordinates for g_0 centered at p, we have that $\mathring{g}_{ij}(0)=h_{ij}$ and the deviations of $\mathring{g}_{ij}$ from the Minkowski metric restricted to $\{x^0=0\}\cap\mathcal{U}$ are quadratic on the coordinates away from the origin. Writing

$$\mathring{g}_{ij}=\varphi(g_0)_{ij}+(1-\varphi)h_{ij}=h_{ij}+\varphi((g_0)_{ij}-h_{ij}),$$

we see that, shrinking $\mathcal{U}$ if necessary and taking into account our choice for $\mathring{g}_{0\alpha}$, $\mathring{g}_{\alpha\beta}$ is a perturbation of the Minkowsi metric restricted to $\{x^0=0\}$. Therefore, $\mathring{g}_{\alpha\beta}$ defines a Lorentzian metric.

Next, we introduce

$$\mathring{u}^i=\varphi v_0^i,\ \partial_0\mathring{u}^i=\varphi v_1^i,$$

with the initial conditions for $\mathring{u}^0$ and $\partial_0\mathring{u}^0$ obtained by the same formulas as in section (3.2), with the appropriate replacements by $\mathring{u}^i$ and $\mathring{g}$ on the right-hand sides. Finally, set

$$\mathring{\epsilon}=\varphi\epsilon_0+1-\varphi,\ \partial_0\mathring{\epsilon}=\varphi\epsilon_1.$$

By the compactness of Σ and the assumption $\epsilon_0>0$, it follows that $\epsilon_0\geq C$ for some constant $C>0$, thus

$$\mathring{\epsilon}\geq\min\{\frac{1}{2}C,\frac{1}{2}\}\geq C'>0,$$

for some constant C'.

The initial data for (9) in $\mathbb{R}^4$ described in this section will be denoted by $\mathring{U}$.

3.4. **Solving the system in $\mathbb{R}^4$.** In this section, we solve system (9) with the initial conditions described in section 3.3 (see Proposition 1 below). We shall employ the techniques, terminology, and notation of Leray-Ohya systems reviewed in the appendix.

Lemma 3.1. *Equations (9) form a Leray system.*

Proof. Write U as $U = (U^1, U^2)$, with the understanding that $U^1 = (u^\beta, \epsilon) = (u^0, u^1, u^2, u^3, \epsilon)$ and $U^2 = (g_{\alpha\beta})$. Assign to (9) the following indices:

$$\begin{aligned} m_1 &= 2, \quad m_2 = 2, \\ n_1 &= 0, \quad n_2 = 0, \end{aligned}$$

where $m_1 = m(U^1) \equiv m(u^\beta, \epsilon)$, $m_2 = m(U^2) \equiv m(g_{\alpha\beta})$,

$$\begin{aligned} n_1 &= n(\text{equation (7)}) \\ &= n(\text{equation (8)}) \\ &\equiv n(\text{equations corresponding to the first five rows of (9)}), \end{aligned}$$

and

$$\begin{aligned} n_2 &= n(\text{equation (6)}) \\ &\equiv n(\text{equations corresponding to the last ten rows of (9)}). \end{aligned}$$

It is understood that we have one index m_I for each unknown of the fifteen unknowns and one index n_J for each one of the fifteen equations in (9). For instance, by $m_1 = m_1(u^\beta, \epsilon) = 2$ we mean $m(u^0) = m(u^1) = m(u^2) = m(u^3) = m(\epsilon) = 2$, and so on.

One readily verifies that with this choice of indices, (9) has the structure of a Leray system. Indeed, we list below for each row J in (9) or, equivalently, for each equation in the system (6), (7), and (8), the value of n_J; the highest derivatives of each unknown entering in the coefficients and on the right-hand side of the equation; and the difference $m_I - n_J$:

$$\text{rows 1-4} \equiv \text{eq. (7)} : n_1 = 0;\ \partial u, \partial\epsilon, \partial g; \begin{cases} m(u) - n_1 \equiv m_1 - n_1 = 2, \\ m(\epsilon) - n_1 \equiv m_1 - n_1 = 2, \\ m(g) - n_1 \equiv m_2 - n_1 = 2, \end{cases}$$

$$\text{row 5} \equiv \text{eq. (8)} : n_1 = 0;\ \partial u, \partial g; \begin{cases} m(u) - n_1 \equiv m_1 - n_1 = 2, \\ m(\epsilon) - n_1 \equiv m_1 - n_1 = 2, \\ m(g) - n_1 \equiv m_2 - n_1 = 2, \end{cases}$$

and

$$\text{rows 6-15} \equiv \text{eq. (6)} : n_2 = 0;\ \partial u, \partial\epsilon, \partial g; \begin{cases} m(u) - n_1 \equiv m_1 - n_2 = 2, \\ m(\epsilon) - n_1 \equiv m_1 - n_2 = 2, \\ m(g) - n_1 \equiv m_2 - n_2 = 2. \end{cases}$$

For example, in equations (7), for which $n_1 = 0$, we have that the left-hand side consists of differential operators of order 2 acting on (u^β, ϵ) $(m(u^\beta, \epsilon) - n_1 = 2)$ and differential operators of order 2 acting on $(g_{\alpha\beta})$ $(m(g_{\alpha\beta}) - n_1 = 2)$, whose coefficients depend on at most first derivatives of the unknowns ($\partial u, \partial\epsilon, \partial g$, i.e., $m(u^\beta, \epsilon) - n_1 - 1$

and $m(g_{\alpha\beta})-n_1-1$); the right-hand side of (7), as the coefficients of the differential operators, depends on at most first derivatives of the unknowns. □

Assumption 1. We henceforth make explicit use of (5), with $a_1 = 4$ and $a_2 \geq 4$, in accordance with the assumptions of Theorem 2.2.

For the proof of the next proposition, the reader is reminded of the Definition A.9 of $\mathcal{A}^s(\Sigma, Y)$, which consists of the space of functions sufficiently near the Cauchy data.

Proposition 1. *There exist a $\mathcal{T} > 0$, a vector field $u : [0,\mathcal{T})\times\mathbb{R}^3 \to \mathbb{R}^4$, a function $\epsilon : [0,\mathcal{T})\times\mathbb{R}^3 \to (0,\infty)$, and a Lorentzian metric g defined on $[0,\mathcal{T})\times\mathbb{R}^3$, such that $U = (u^\beta, \epsilon, g_{\alpha\beta})$ satisfies (9) in $[0,\mathcal{T})\times\mathbb{R}^3$ and takes the initial data $\mathring{U}$ on $\{x^0 = 0\}$. Moreover, $(u,\epsilon,g) \in G^{2,(s)}([0,\mathcal{T})\times\mathbb{R}^3)$ and this solution is unique in this class.*

Proof. We fix the initial data $\mathring{U}$ as constructed in section 3.3 and consider $\widehat{U} = (\widehat{u}^\alpha, \widehat{\epsilon}, \widehat{g}_{\alpha\beta}) \in \mathcal{A}^s(\Sigma, Y)$. Shrinking Y if necessary, we can assume that $\widehat{g}_{\alpha\beta}$ is a Lorentzian metric, that $\widehat{\epsilon} > 0$, and that $\widehat{u}$ is time-like for $\widehat{g}_{\alpha\beta}$, since these properties hold for $\mathring{U}$. Because the coefficients of the matrix of differential operators $\mathfrak{M}(U,\partial)$ depend on at most first derivatives of the unknowns, we can evaluate these coefficients on $\widehat{U}$. Denote the corresponding operator by $\mathfrak{M}(\widehat{U},\partial)$. The characteristic determinant $P(\widehat{U},\xi)$ of (9), evaluated at $\widehat{U}$, is

$$P(\widehat{U},\xi) = \det \mathfrak{M}(\widehat{U},\xi) = p_1(\widehat{U},\xi)p_2(\widehat{U},\xi)p_3(\widehat{U},\xi)p_4(\widehat{U},\xi) \tag{11}$$

where[7]

$$p_1(\widehat{U},\xi) \equiv p_1(\xi) = \frac{1}{12\widehat{\epsilon}}\eta^4(\widehat{u}^\mu\xi_\mu)^4, \tag{12}$$

$$\begin{aligned} p_2(\widehat{U},\xi) \equiv p_2(\xi) = \big[&(a_2-1)((\widehat{u}^0)^2\xi_0^2 + (\widehat{u}^1)^2\xi_1^2 + (\widehat{u}^2)^2\xi_2^2 + (\widehat{u}^3)^2\xi_3^2) - \xi^\mu\xi_\mu \\ &+2(a_2-1)(\widehat{u}^1\widehat{u}^2\xi_1\xi_2 + \widehat{u}^1\widehat{u}^3\xi_1\xi_3 + \widehat{u}^2\widehat{u}^3\xi_2\xi_3) \\ &+2(a_2-1)\widehat{u}^0\xi_0\widehat{u}^i\xi_i\big]^2, \end{aligned} \tag{13}$$

$$\begin{aligned} p_3(\widehat{U},\xi) \equiv p_3(\xi) = &-6((a_2+5)a_2 + (a_2^2+7a_2-8)\widehat{u}^\lambda\widehat{u}_\lambda)(\widehat{u}^\mu\xi_\mu)^2 \\ &+6(a_2+2)(1+5\widehat{u}^\lambda\widehat{u}_\lambda)\xi^\mu\xi_\mu, \end{aligned} \tag{14}$$

and

$$p_4(\widehat{U},\xi) \equiv p_4(\xi) = (\xi^\mu\xi_\mu)^{10}, \tag{15}$$

and the contractions in these expressions are done with respect to the metric $\widehat{g}_{\alpha\beta}$. The computation of $P(\widehat{U},\xi)$, and the corresponding factorization in the above polynomials, is done through a lengthy and tedious algebraic calculation, part of which was done with the help of the software Mathematica[8]. Note that the block diagonal form of $\mathfrak{M}(U,\partial)$ allowed us to compute the characteristic determinant without providing the specific form of the operators $\widetilde{B}^\beta(\partial u, g)\partial^2 g$ and $\widetilde{B}(\partial u, g)\partial^2 g$.

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[7] We remark that compared to [3], polynomial $p_3(\widehat{U},\xi)$ looks different. That is because in [3] $\widehat{u}^\lambda\widehat{u}_\lambda$ had been replaced by -1 in view of (2). Strictly speaking, we are not allowed to do that since one has to prove that u remains normalized for positive time, which is done in Lemma 3.3 below, but this was ignored in [3] since there only a sketch of the proof was presented (see the above Introduction).

[8] See Appendix C.

It is easy to see that the polynomials $\widehat{u}^\mu \xi_\mu$ and $\xi^\mu \xi_\mu$ are hyperbolic polynomials as long as $\widehat{g}_{\alpha\beta}$ is a Lorentzian metric and $\widehat{u}$ is time-like with respect to $\widehat{g}_{\alpha\beta}$. Both conditions are satisfied in view of the constructions in section 3.3. Therefore, $p_1(\xi)$ is the product of four hyperbolic polynomials (recall that $\widehat{\epsilon} > 0$ and $\eta(\widehat{\epsilon}) > 0$), and $p_4(\xi)$ is the product of ten hyperbolic polynomials. We now move to analyze $p_2(\xi)$ and $p_3(\xi)$.

Write $p_2(\xi) = (\widetilde{p}_2(\xi))^2$, where $\widetilde{p}_2(\xi)$ is the second-degree polynomial between brackets in the definition of $p_2(\xi)$. We claim that $\widetilde{p}_2(\xi)$ is a hyperbolic polynomial. To show this, we need to investigate the roots $\xi_0 = \xi_0(\xi_1, \xi_2, \xi_3)$ of the equation $\widetilde{p}_2(\xi) = 0$. Consider first the case where $\widetilde{p}_2(\xi)$ is evaluated at the origin, i.e., $\widetilde{p}_2(\xi) = \widetilde{p}_2(\widehat{U}(0), \xi)$, and assume for a moment that $\widehat{g}_{\alpha\beta}(0)$ is the Minkowski metric and that $\widehat{u}^\mu \widehat{u}_\mu = -1$. In this case, the roots are

$$\begin{aligned} \xi_{0,\pm} = -\frac{1}{1 + (a_2 - 1)(1 + \underline{\widehat{u}}^2)} \Bigg(&(a_2 - 1)\underline{\widehat{u}} \cdot \underline{\xi} \sqrt{1 + \underline{\widehat{u}}^2} \\ &\pm \sqrt{(a_2 + (a_2 - 1)\underline{\widehat{u}}^2)\underline{\xi}^2 - (a_2 - 1)(\underline{\widehat{u}} \cdot \underline{\xi})^2} \Bigg), \end{aligned} \tag{16}$$

where $\underline{\widehat{u}} = (\widehat{u}^1, \widehat{u}^2, \widehat{u}^3)$, $\underline{\widehat{u}}^2 = (\widehat{u}^1)^2 + (\widehat{u}^2)^2 + (\widehat{u}^3)^2$, $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$, $\underline{\xi}^2 = \xi_1^1 + \xi_2^2 + \xi_3^2$, and $\cdot$ is the Euclidean inner product. We see that if $\underline{\xi} = 0$, then $\xi_{0,\pm} = 0$, and hence $\xi = 0$. Thus, we can assume $\underline{\xi} \neq 0$. The Cauchy-Schwarz inequality gives $\underline{\widehat{u}}^2 \underline{\xi}^2 - (\underline{\widehat{u}} \cdot \underline{\xi})^2 \geq 0$, hence $\xi_{0,+}$ and $\xi_{0,-}$ are real and distinct for $a_2 \geq 4$. We conclude that $\widetilde{p}_2(\xi)$ is a hyperbolic polynomial at the origin. Since the roots of a polynomial vary continuously with the polynomial coefficient, $\widetilde{p}_2(\xi)$ will have two distinct real roots at any point on $\{x^0 = 0\}$ if $\widehat{g}_{\alpha\beta}$ is sufficiently close to the Minkowski metric and $\widehat{u}^\mu \widehat{u}_\mu$ sufficiently close to -1. We know from section 3.3 that these last conditions are fulfilled upon taking $\mathcal{U}$ and Y sufficiently small (recall that $\mathring{g}_{\alpha\beta}(0)$ equals the Minkowski metric.). Therefore, $\widetilde{p}_2(\xi)$ is a hyperbolic polynomial, and $p_2(\xi)$ is the product of two hyperbolic polynomials.

We now investigate the roots $\xi_0 = \xi_0(\xi_1, \xi_2, \xi_3)$ of the equation $p_3(\xi) = 0$. As above, we first consider $p_3(\xi)$ evaluated at the origin and suppose that $\widehat{g}_{\alpha\beta}(0)$ is the Minkowski metric and that $\widehat{u}^\mu \widehat{u}_\mu = -1$, which produces

$$\begin{aligned} \xi_{0,\pm} = \frac{1}{-2(2 + a_2) - (a_2 - 4)(1 + \underline{\widehat{u}}^2)} \Bigg(&(a_2 - 4)\underline{\widehat{u}} \cdot \underline{\xi} \sqrt{1 + \underline{\widehat{u}}^2} \\ &\pm \sqrt{2} \sqrt{(3a_2(2 + a_2) + (a_2^2 - 2a_2 - 8)\underline{\widehat{u}}^2)\underline{\xi}^2 - (a_2^2 - 2a_2 - 8)(\underline{\widehat{u}} \cdot \underline{\xi})^2} \Bigg). \end{aligned}$$

As above, we can assume $\underline{\xi} \neq 0$, and the Cauchy-Schwarz inequality again gives $\underline{\widehat{u}}^2 \underline{\xi}^2 - (\underline{\widehat{u}} \cdot \underline{\xi})^2 \geq 0$. We readily verify that $(a_2^2 - 2a_2 - 8) \geq 0$ and $3a_2(2 + a_2) > 0$ for $a_2 \geq 4$. Therefore, $\xi_{0,+}$ and $\xi_{0,-}$ are real and distinct, and $p_3(\xi)$ is a hyperbolic polynomial at the origin. As above, this implies that $p_3(\xi)$ is a hyperbolic polynomial.

We conclude that $P(\widehat{U}, \xi)$ is the product of four degree one (i.e., $p_1(\xi)$), two degree two (i.e., $p_2(\xi)$), one degree two (i.e., $p_3(\xi)$), and ten degree two (i.e., $p_4(\xi)$) hyperbolic polynomials. The Gevrey index of (9) is thus $\frac{17}{16}$ (see Remark 15). Recall that $1 < s < \frac{17}{16}$ by assumption.

Since $m_I - n_J = 2$ for all I, J, and $\sum_I m_I - \sum_J n_J \geq 2$, we have verified the conditions of Theorem A.14 in the appendix. Hence we obtain the diagonalized system

$$\widetilde{\mathfrak{M}}(U,\partial)U = \widetilde{\mathfrak{q}}(U), \tag{17}$$

where $\widetilde{\mathfrak{M}}(U,\partial)$ is a diagonal matrix whose entries are differential operators of order 30 (the order of the characteristic determinant, see the appendix) whose coefficients depend on at most 29 derivatives of U, and $\widetilde{\mathfrak{q}}(U)$ contains all the lower order terms. We want to invoke Theorem A.10 to solve (17). To do so, we need to provide initial conditions for (17). Since our goal is to obtain a solution to (9) out of a solution to (17), such initial conditions need to be compatible with solutions to (9).

We shall show that all derivatives of U, restricted to $\{x^0 = 0\}$, can be formally computed from (9) and written in terms of the initial data. In particular, initial conditions to (17) compatible with (9) can be determined. As usual in these situations, it suffices to show that we can inductively compute $\partial_0^k U$ on $\{x^0 = 0\}$ as the tangential derivatives ∂_i can always be computed.

From (6), we can determine $\partial_0^2 g_{\alpha\beta}\big|_{\{x^0=0\}}$ in terms of the initial data $\mathring{U}$. Using the result into (7), we can write $\widetilde{B}^\beta(\partial u, g)\partial^2 g$ restricted to $\{x^0 = 0\}$ in terms of $\mathring{U}$. Equations (7) and (8) then give

$$\mathfrak{a}\begin{pmatrix} \partial_0^2 u^\beta \\ \partial_0^2 \epsilon \end{pmatrix} = \mathfrak{b},$$

where $\mathfrak{b}$ can be written in terms of the initial data on $\{x^0 = 0\}$, and the matrix $\mathfrak{a}$ is the matrix of the coefficients of the terms $\partial_0^2 u^\beta$ and $\partial_0^2\epsilon$ in equations (7) and (8). At the origin, where $\mathring{g}_{\alpha\beta}(0)$ equals the Minkowski metric, the determinant of $\mathfrak{a}$ is

$$\frac{\eta^4}{\epsilon_0}(1+\underline{\mathring{u}}^2)^2(3a_2 + (a_2-4)\underline{\mathring{u}}^2)(a_2 + (a_2-1)\underline{\mathring{u}}^2)^2,$$

which is never zero for $a_2 \geq 4$ (recall that $\epsilon_0 > 0$ and $\eta(\epsilon_0) > 0$). Invoking once more the fact that $\mathring{g}_{\alpha\beta}$ is a perturbation of the Minkowski metric, we conclude that $\det(\mathfrak{a})\big|_{\{x^0=0\}}$ never vanishes. We can thus invert $\mathfrak{a}$ and write $\partial_0^2 u^\beta$ and $\partial_0^2\epsilon$ at $x^0 = 0$ in terms of $\mathring{U}$.

It is clear that we can continue this process: differentiate (6) with respect to ∂_0 to determine $\partial_0^3 g_{\alpha\beta}\big|_{\{x^0=0\}}$; differentiate (7) and (8) with respect to ∂_0, use $\partial_0^3 g_{\alpha\beta}\big|_{\{x^0=0\}}$ to eliminate the resulting terms $\widetilde{B}^\beta(\partial u, g)\partial^3 g$ and $\widetilde{B}(\partial u, g)\partial^3 g$, and then solve for $\partial_0^3 u^\beta$ and $\partial_0^3\epsilon$ at $x^0 = 0$ (notice that the matrix $\mathfrak{a}$ remains unchanged). Inductively, we can determine all derivatives $\partial_0^k U$ on $\{x^0 = 0\}$, $k = 2, 3, \dots,$ in terms of $\mathring{U}$. Moreover, $\partial_0^k U\big|_{\{x^0=0\}}$ are analytic expressions of $\mathring{U}$ and, therefore, the initial conditions for (17) determined in this fashion will be in $G^{(s)}$.

The initial data for (17), denoted $\mathring{\mathring{U}}$, consists of the original initial data $\mathring{U}$ for (9), and the values of $\partial_0^k U\big|_{\{x^0=0\}}$ determined by the above procedure for $k = 2, \dots, 29$.

Remark 2. The above procedure determines all derivatives of U, evaluated at $x^0 = 0$, in terms of the initial conditions $\mathring{U}$. It follows that if the initial data $\mathring{U}$ is analytic, a well-known argument using power series can be employed to construct an analytic solution to (9) in a neighborhood of $\{x^0 = 0\}$. These techniques for construction of analytic solutions, however, say nothing about causality.

Having supplied (17) with appropriate initial conditions, we can now invoke Theorem A.10 to conclude the following. There exist a $\widetilde{\mathcal{T}} > 0$, a vector field

$u : [0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3 \to \mathbb{R}^4$, a function $\epsilon : [0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3 \to (0, \infty)$, and a Lorentzian metric g defined on $[0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3$, such that $U = (u^\beta, \epsilon, g_{\alpha\beta})$ satisfies (17) in $[0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3$ and takes the initial data $\mathring{\widetilde{U}}$ on $\{x^0 = 0\}$. Moreover, $(u, \epsilon, g) \in G^{2,(s)}([0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3)$ and this solution is unique in this class.

(We note that in invoking Theorem A.10, we are using that the intersections of the cones determined by the polynomials $p_i(\xi)$ have non-empty interiors (recall definition A.4). This follows from the above expressions, but it can also be verified from the explicit computations in section 3.5.)

The conclusions that $\epsilon > 0$ and g is a Lorentzian metric follow by continuity in the x^0 variable, since these conditions are true at $x^0 = 0$.

Now we move to obtain a solution to (9) in $\mathbb{R}^4$. The argument is similar to the one in [30], thus we shall go over it briefly.

Let $\{\mathring{U}_k\}_{k=1}^\infty$ be a sequence of analytic initial conditions for the system (9) converging in $G^{(s)}(\{x^0 = 0\})$ to $\mathring{U}$. For each k, let (u_k, ϵ_k, g_k) be the analytic solution to (9), defined in a neighborhood of $\{x^0 = 0\}$, and taking on the initial data $\mathring{U}_k$ (see Remark 2). Let $\mathring{\widetilde{U}}_k$ be the initial data for (17) obtained from $\mathring{U}_k$ and compatible with (9), i.e., the one derived by the inductive procedure previously described. Then, $\mathring{\widetilde{U}}_k \to \mathring{\widetilde{U}}$ in $G^{(s)}(\{x^0 = 0\})$. In light of the compatibility of $\mathring{\widetilde{U}}_k$, and because (17) was derived from (9) via diagonalization, the solutions (u_k, ϵ_k, g_k) also satisfy (17). Furthermore, this solution to (17) also agrees with the one given by Theorem A.10 (since this theorem also applies for analytic data, i.e., $s = 1$). The energy-type of estimates proved by Leray and Ohya [28] guarantee then that $(u_k, \epsilon_k, g_k) \to (u, \epsilon, g)$ in $G^{(s)}$ and that (u, ϵ, g) satisfy the original system (9). By construction, (u, ϵ, g) take on the initial data $\mathring{U}$. $\square$

Remark 3. The initial conditions for the VECF system have to satisfy the Einstein constraint equations (recall Definition 2.1). The initial conditions $\mathring{U}$ satisfy the constraints in the region $\mathcal{W}$ in light of the way that $\mathring{U}$ was constructed out of $\mathcal{I}|_{\mathcal{U}}$. This is, naturally, necessary for the eventual construction of a full solution to the VECF system. However, purely from the point of view of (9) in $\mathbb{R}^4$, initial condition can be prescribed freely, i.e., they do not have to satisfy any constraints. Therefore, the existence of the analytic initial data $\mathring{U}_k$ follows simply by the density of analytic functions in $G^{(s)}$. Also by density, we can guarantee that the components $(\mathring{\epsilon}_0)_k$ and $(\mathring{g}_{\alpha\beta})_k$ in $\mathring{U}_k$ satisfy $(\mathring{\epsilon}_0)_k > 0$ and that $(\mathring{g}_{\alpha\beta})_k$ is a Lorentzian metric.

Remark 4. The above calculations involving $(a_2^2 - 2a_2 - 8) \geq 0$ show why we have the technical assumption $a_2 \geq 4$. As our calculations were presented already with $a_1 = 4$ in place, they do not reveal the reason for this assumption, which as follows. Computing the characteristic determinant with general a_1 produces a very complicated expression with some terms proportional to $a_1 - 4$. These terms vanish when $a_1 = 4$, and the corresponding expression simplifies to (11). This can be seen explicitly in Appendix C.

3.5. **Causality.** Having obtained solutions, we now investigate the causality of equations (9). As in section 3.4, we use results and terminology recalled in the appendix.

Lemma 3.2. *The solution $U = (u, \epsilon, g)$ to (9) given in Proposition 1 is causal, in the following sense. For any $x \in [0, \mathcal{T}) \times \mathbb{R}^3$, $(u(x), \epsilon(x), g(x))$ depends only on $\mathring{U}|_{\{x^0=0\} \cap J^-(x)}$, where $J^-(x)$ is the causal past of x (with respect to the metric g).*

Proof. Fix $x \in [0, \mathcal{T}) \times \mathbb{R}^3$. The characteristic determinant of (9) at x is given by (11), with the obvious replacement of $\widehat{U}$ by U and evaluated at x; the polynomials $p_i(U(x), \xi) \equiv p_i(x, \xi)$, $i = 1, \dots, 4$, are given by expressions (12) to (15), again with the obvious replacement by $U(x)$. By the same argument used in section 3.4 to prove that the $p_i(\xi)$'s are hyperbolic polynomials on $\{x^0 = 0\}$, namely, that $g_{\alpha\beta}$ is near the Minkowski metric, we know that the polynomials $p_i(x, \xi)$ are hyperbolic (perhaps after shrinking $\mathcal{T}$ if necessary).

Denote by $V_i(x)$ the characteristic cone $\{p_i(x, \xi) = 0\}$, and by $\Gamma_i^{*,\pm}(x)$ the corresponding (forward and backward) convex cones (on the cotangent space). Let $K^{*,\pm}(x)$ be the (forward and backward) time-like interiors of the light-cone $\{g^{\mu\nu}(x)\xi_\mu\xi_\nu = 0\}$. We need to show that $K^{*,\pm}(x) \subseteq \Gamma_i^{*,\pm}(x)$ (see Remark 12). This is straightforward for $i = 1$ and $i = 4$.

Assume for a moment that g is the Minkowski metric at x and that $u^\lambda u_\lambda = -1$ (note that we have not proved yet that u remains normalized for $x^0 > 0$). The roots of $\{p_2(x, \xi) = 0\}$ are given by (16), changing $\widehat{u}$ by u, which we can write as

$$\xi_{0,\pm} = s_\pm(u, \theta)\sqrt{\underline{\xi}^2}, \tag{18}$$

where

$$s_\pm(u,\theta) = -\frac{1}{1 + (a_2 - 1)(1 + \underline{u}^2)}\Big((a_2 - 1)\sqrt{\underline{u}^2}\cos\theta\sqrt{1 + \underline{u}^2} \\ \pm\sqrt{a_2 + (a_2 - 1)\underline{u}^2 - (a_2 - 1)\underline{u}^2\cos^2\theta}\Big),$$

θ is the angle between $\underline{u}$ and $\underline{\xi}$ in $\mathbb{R}^3$, we used $\underline{u}\cdot\underline{\xi} = \sqrt{\underline{u}^2}\sqrt{\underline{\xi}^2}\cos\theta$, and we omitted the dependence of u and θ on x for simplicity.

Equation (18) determines the two halves of the characteristic cone $V_2(x)$ in the cotangent space at x. We will have that $K^{*,\pm}(x) \subseteq \Gamma_2^{*,\pm}(x)$ if the slopes $s_\pm$ satisfy $-1 < s_\pm(u, \theta) < 1$ for each u and θ. To see that this is the case, compute

$$s_\pm(u, 0) = s_\pm(u, 2\pi) = -\frac{\pm\sqrt{a_2} + (a_2 - 1)\sqrt{\underline{u}^2(1 + \underline{u}^2)}}{1 + (a_2 - 1)(1 + \underline{u}^2)},$$

and observe that this expression is always between -1 and 1 for $a_2 \geq 4$. We seek the maxima and minima of $s_\pm(u, \theta)$ for $0 < \theta < 2\pi$. Computing the derivative with respect to θ and solving for $\sin\theta$, we find $\sin\theta = 0$, i.e., $\theta = \pi$. We readily verify that $-1 < s_\pm(u, \pi) < 1$, thus $-1 < s_\pm(u, \theta) < 1$. Since this last condition is open, the result remains true when g is sufficiently close to the Minkowski metric and u sufficiently close to unitary, which is the case if $\mathcal{T}$ is taken sufficiently small. The same argument shows that $K^{*,\pm}(x) \subseteq \Gamma_3^{*,\pm}(x)$, where again one uses the condition $a_2 \geq 4$.

We conclude that for any $x \in [0, \mathcal{T}) \times \mathbb{R}^3$, we have $K^{*,\pm}(x) \subseteq \bigcap_{i=1}^4 \Gamma_i^{*,\pm}(x)$, and the result now follows from Theorem A.11 and Remark 12. $\square$

Remark 5. The characteristics associated with $p_1(\xi)$ and $p_4(\xi)$ are of course those of the flow lines and gravitational waves. The characteristics associated with $p_3(\xi)$ and $p_2(\xi)$ are interpreted, respectively, as sound waves and shear waves. The latter is sometimes called a second sound wave and is present also in the Müller-Israel-Stewart theory [22]. It is useful to compare these characteristics to those of the ideal fluid. In the latter case we have the flow lines and the sound cone (i.e., the characteristics of the sound waves; see [17] for a detailed discussion of the role of the sound cone in the relativistic Euler equations). Here it is as if the sound cone had "split" into two sound-type characteristics. This resembles what happens in magnetohydrodynamics: there two different characteristics are present for the magnetoacoustic waves, namely, the so-called fast and slow magnetoacoustic waves (see [1] for details).

3.6. **Existence and causality for the system in $\mathbb{R}\times\Sigma$.** Here we show how the solution found in section 3.4 can be used to construct a causal solution in a region of $\mathbb{R}\times\Sigma$, thus effectively proving Theorem 2.2. Recall that we embedded Σ into $\mathbb{R}\times\Sigma$.

Remark 6. Consider the solution $U=(u,\epsilon,g)$ to (9) obtained in Proposition 1. Let p be a point on $\{x^0=0\}\times\Sigma$ and $\mathcal{W}$ be as in section 3.3. Let $D_g^+(\mathcal{W})\subseteq[0,\mathcal{T})\times\mathbb{R}^3$ be the future domain of dependence of $\mathcal{W}$ in the metric g, where replacing $\mathcal{W}$ with a smaller set if necessary, we can assume that $x^0<\mathcal{T}$ for every $(x^0,x^1,x^2,x^3)\in D_g^+(\mathcal{W})$. In the coordinates on $D_g^+(\mathcal{W})$ induced from the coordinates on $[0,\mathcal{T})\times\mathcal{W}$, the solution U is in $G^{(2,s)}$ The solution will remain in $G^{(2,s)}$ upon coordinate changes that are Gevrey regular [32]. Note that there are plenty of such coordinate changes in that a smooth manifold always admits a maximal compatible analytic atlas.

Lemma 3.3. *It holds that $u^\lambda u_\lambda=-1$ in $D_g^+(\mathcal{W})$.*

Proof. The vector field u satisfies (8), whose explicit form is

$$u_\lambda u^\alpha u^\mu\nabla_\mu\nabla_\mu u^\lambda+u^\alpha\nabla_\alpha u_\lambda u^\mu\nabla_\mu u^\lambda=0.$$

This can be written as

$$\frac{1}{2}u^\alpha u^\mu\nabla_\alpha\nabla_\mu(u_\lambda u^\lambda)=0.$$

This is an equation for the scalar $u_\lambda u^\lambda$. The operator $u^\alpha u^\mu\nabla_\alpha\nabla_\mu$ satisfies the assumptions of Theorem A.10. Therefore, $u_\alpha u^\alpha=-1$ in $D_g^+(\mathcal{W})$ if this condition is satisfied initially, which is the case by construction. $\square$

Lemma 3.4. *For every $q\in\Sigma$ there exists a neighborhood $Z_q\subseteq\Sigma$ of q in Σ and a globally hyperbolic development M_q of $\mathcal{I}|_{Z_q}$, where $M_q\subseteq[0,\mathcal{T}_q)\times\Sigma$ for some $\mathcal{T}_q>0$.*

Proof. Let p be a point on $\{x^0=0\}\times\Sigma$ and $\mathcal{W}$ be as in section 3.3. Since the initial conditions $\mathring{U}$ (where $\mathring{U}$ is as in section 3.3) agree on $\mathcal{W}$ with those from the initial data $\mathcal{I}$, in view of Lemma 3.2, we conclude that U is a solution to the reduced Einstein equations within $D_g^+(\mathcal{W})$. It is well-known that a solution to the reduced equations within $D_g^+(\mathcal{W})$ is also a solution to the full Einstein's equations if and only if the constraints are satisfied, which is the case by the definition of $\mathcal{I}$. Because p was an arbitrary point, the result is proven. $\square$

We now glue the different M_q's in order to obtain a global (in space) solution.

Proposition 2. *Let $q,r\in\Sigma$, Z_q and Z_r be neighborhoods of q and r as in lemma 3.4, with globally hyperbolic developments M_q and M_r of $\mathcal{I}|_{Z_q}$ and $\mathcal{I}|_{Z_r}$, respectively, and corresponding solutions $U_q=(u_q,\epsilon_q,g_q)$ and $U_r=(u_r,\epsilon_r,g_r)$ of the VECF equations. Assume that $Z_q\cap Z_r\neq\varnothing$. Then, for any $w\in Z_q\cap Z_r$, there exist neighborhoods $\mathcal{U}_q$ and $\mathcal{U}_r$ of w in M_q and M_r, respectively, and a diffeomorphism $\psi:\mathcal{U}_q\to\mathcal{U}_r$ such that $U_q=\psi^*(U_r)$.*

Proof. We shall construct harmonic coordinates for g_q in a neighborhood of w in M_q as follows. Identifying (a portion of) Σ with its embedding in M_q, take normal coordinates $(V, \{y^i\})$ for g_0 on Σ centered at w, where g_0 comes from the initial data $\mathcal{I}$. Note that the initial data is Gevrey regular in the $\{y^i\}$ coordinates (see the argument in section 3.2). We can thus assume that U_q is in $G^{(2,s)}$ (see Remark 6)

On $[0, \mathcal{T}_q) \times V$, where $\mathcal{T}_q > 0$ is some small number such that U_q is defined on $[0, \mathcal{T}_q) \times V$, we introduce coordinates $\{y^\alpha\}$, $y^0 \in [0, \infty)$. Consider family of initial-value problems parametrized by α:

$$
\begin{aligned}
\nabla^\mu \nabla_\mu f^{(i)} &= 0, \\
f^{(i)}(0, y^1, y^2, y^3) &= y^i, \\
\partial_0 f^{(i)}(0, y^1, y^2, y^3) &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla^\mu \nabla_\mu f^{(0)} &= 0, \\
f^{(0)}(0, y^1, y^2, y^3) &= 0, \\
\partial_0 f^{(0)}(0, y^1, y^2, y^3) &= 1,
\end{aligned}
$$

where ∇ is the covariant derivative in the metric g_q. This problem has a Gevrey regular solution in a neighborhood of w in $[0, \mathcal{T}_q) \times V$, and a standard implicit function type of argument shows that the functions $x^\alpha \equiv f^{(\alpha)}$ define (harmonic) coordinates near w. We now consider the change of coordinates $x = x(y) : [0, \mathcal{T}_q') \times V' \to W \subseteq [0, \infty) \times \mathbb{R}^3$, $x = (x^0, x^1, x^2, x^3)$, where V' is a neighborhood of w in V, $\mathcal{T}' > 0$ is determined by the foregoing conditions guaranteeing the existence of the coordinates $\{x^\alpha\}$, and W is an open set containing the origin. Pulling U_q back to W via x^{-1}, it follows from these constructions that $(x^{-1})^*(U_q)$ satisfies the reduced Einstein equations in W. Since U_q originally satisfied (2) and (4) as well, we conclude that it is a solution to (9) in W.

We can repeat the above argument to obtain wave coordinates $\{z^\alpha\}$ for g_r. Because $(V, \{y^i\})$ is intrinsically determined by g_0, and M_q and M_r induce on $Z_q \cap Z_r$ the same initial data, the map z agrees with x on $\{0\} \times V'$ (in the region where both are defined). From these facts, we conclude that $(x^{-1})^*(U_q)$ and $(z^{-1})^*(U_r)$ (i) are solutions to (9) in some domain $[0, t) \times Y \subseteq [0, \infty) \times \mathbb{R}^3$ containing the origin, and (ii) take the same initial data on $\{0\} \times Y$.

We have shown that (9) enjoys uniqueness and causality. Thus, considering possibly a smaller region that is globally hyperbolic for both $(x^{-1})^*(g_q)$ and $(z^{-1})^*(g_r)$, we conclude that $(x^{-1})^*(U_q) = (z^{-1})^*(U_r)$, so that $U_q = (z^{-1} \circ x)^*(U_r)$, as desired. $\square$

Using Proposition 2, we can now identify overlapping globally hyperbolic developments, thus obtaining a globally hyperbolic development of $\mathcal{I}$ as stated in Theorem 2.2. Causality follows essentially from Lemma 3.2: by the foregoing, we can assume that M is diffeomorphic to $[0, \mathcal{T}) \times \Sigma$ for some $\mathcal{T} > 0$. Shrinking $\mathcal{T}$ if necessary, we reduce the problem to local coordinates, in which case we can employ wave coordinates. Causality, as stated in Theorem 2.2, is preserved by diffeomorphisms, thus the result follows from the causality of the reduced system guaranteed by Lemma 3.2. This finishes the proof of Theorem 2.2.

4. **Proof of Theorem 2.3.** The proof of Theorem 2.3 is essentially contained in the above. In the case of a Minkowski background, the system reduces to

$$m(U, \partial)U = \mathfrak{q}(U),$$

where m is as in (10), $U = (u^\beta, \epsilon)$ and $\mathfrak{q}(U)$ is as in (9) with the appropriate changes for this 5×5 system. The system can be analyzed as in section 3.4. We can do this directly in $\mathbb{R}^4$, without the complications of constructing the initial data $\mathring{U}$. The characteristic determinant is given by $p_1(\xi)p_2(\xi)p_3(\xi)$, where these polynomials are as before, with the simplification that now we need not carry out any near-Minkowski arguments. Without the matrix $g^{\mu\nu}\partial^2_{\mu\nu}$ coming from Einstein's equations, the Gevrey index of the system is $\frac{7}{6}$, and analogues of Proposition 1 and Lemma 3.2 establish the result.

Appendix A. **Tools of weakly hyperbolic systems.** For the reader's convenience, we state in this appendix the results about Leray-Ohya systems (sometimes called weakly hyperbolic systems) that are used in the proof of Theorem 2.2. These results have been established by Leray and Ohya in [27, 28] for the case of systems with diagonal principal part, and extended by Choquet-Bruhat in [6] to more general systems. These works build upon the classical work of Leray on hyperbolic differential equations [26]. The reader can consult these references for the proofs of the results stated below. Further discussion can be found (without proofs) in [7, 10, 12]. Related results can also be found in [34].

We start by recalling some standard notions and fixing the notation that will be used throughout. Given $T > 0$, let $X = [0, T] \times \mathbb{R}^n$. By ∂^k we shall denote any k^{th} order derivative. We shall denote coordinates on X by $\{x^\alpha\}_{\alpha=0}^n$, thinking of $x^0 \equiv t$ as the time-variable. We use the multi-index notation to write

$$\partial^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0}\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} \equiv \partial^{\alpha_0}_{x^0}\partial^{\alpha_1}_{x^1}\partial^{\alpha_2}_{x^2}\cdots\partial^{\alpha_n}_{x^n},$$

where $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

A.1. **Gevrey spaces.** In this section we review the definition of Gevrey spaces. Roughly speaking, a function is of Gevrey class if it obeys inequalities similar, albeit weaker, than those satisfied by analytic functions. One of the crucial properties of Gevrey spaces for their use in general relativity is that they admit compactly supported functions.

Definition A.1. Let $s \geq 1$. We say that $f : \mathbb{R}^n \to \mathbb{C}$ belongs to the Gevrey space $G^{(s)}(\mathbb{R}^n)$ if

$$\sup_\alpha \frac{1}{(1+|\alpha|)^s} \|\partial^\alpha f\|^{\frac{1}{1+|\alpha|}}_{L^2(\mathbb{R}^n)} < \infty.$$

Let $K \subset \mathbb{R}^n$ be the cube of unit side. We say that f belongs to the local Gevrey space $G^{(s)}_{loc}(\mathbb{R}^n)$ if

$$\sup_\alpha \frac{1}{(1+|\alpha|)^s} \left(\sup_K \|\partial^\alpha f\|_{L^2(K)} \right)^{\frac{1}{1+|\alpha|}} < \infty,$$

where $\sup_K$ is taken over all side one cubes K in $\mathbb{R}^n$.

We note that the case $s = 1$, i.e., $G^{(1)}(\mathbb{R}^n)$, corresponds to the space of analytic functions.

We next introduce the space of maps defined on X whose derivatives up to order m belong to $G^{(s)}(\{x^0 = t\})$, $0 \leq t \leq T$.

Definition A.2. On X, denote $S_t = \{x^0 = t\}$. Let $s \geq 1$, and let $m \geq 0$ be an integer. We denote by $\overline{\alpha}$ a multi-index $\alpha = (\alpha_0, \dots, \alpha_n)$ for which $\alpha_0 = 0$. We define $G^{m,(s)}(X)$ as the set of maps $f : X \to \mathbb{C}$ such that

$$\sup_{\overline{\alpha},\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1 + |\overline{\alpha}|)^s} \left\| \partial^{\beta + \overline{\alpha}} f \right\|_{L^2(S_t)}^{\frac{1}{1+|\overline{\alpha}|}} < \infty.$$

Let Y be an open set of $\mathbb{R}^d$. We define $G^{m,(s)}(X \times Y)$ as the set of maps $f : X \times Y \to \mathbb{C}$ such that

$$\sup_{\overline{\alpha},\, \gamma,\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1 + |\overline{\alpha}| + |\gamma|)^s} \left\| \sup_{y \in Y} \left| \partial_x^{\beta + \overline{\alpha}} \partial_y^{\gamma} f \right| \right\|_{L^2(S_t)}^{\frac{1}{1+|\overline{\alpha}|+|\gamma|}} < \infty.$$

Let $K_t \subset S_t$ be the cube whose sides have unit length. The spaces $G^{m,(s)}_{loc}(X)$ and $G^{m,(s)}_{loc}(X \times Y)$ are defined as the set of maps $f : X \to \mathbb{C}$ and $f : X \times Y \to \mathbb{C}$, respectively, such that

$$\sup_{\overline{\alpha},\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1 + |\overline{\alpha}|)^s} \left(\sup_{K_t} \left\| \partial^{\beta + \overline{\alpha}} f \right\|_{L^2(K_t)} \right)^{\frac{1}{1+|\overline{\alpha}|}} < \infty,$$

and

$$\sup_{\overline{\alpha},\, \gamma,\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1 + |\overline{\alpha}| + |\gamma|)^s} \left(\sup_{K_t} \left\| \sup_{y \in Y} \left| \partial_x^{\beta + \overline{\alpha}} \partial_y^{\gamma} f \right| \right\|_{L^2(K_t)} \right)^{\frac{1}{1+|\overline{\alpha}|+|\gamma|}} < \infty,$$

where $\sup_{K_t}$ is taken over all cubes of side one within S_t.

Remark 7. Definitions A.1 and A.2 are easily generalized to vector and tensor fields in $\mathbb{R}^n$ and X, and to open subsets of $\mathbb{R}^n$ and X. In particular, replacing $\mathbb{R}^n$ by an open set Ω and X by $[0, T] \times \Omega$ in the above definitions we obtain the corresponding spaces for Ω. This allows one to define Gevrey spaces on manifolds. If M is a differentiable manifold, we say that $f : M \to \mathbb{C}$ belongs to $G^{(s)}(M)$ if for every $p \in M$ there exists a coordinate chart (x, U) about p such that $f \circ x^{-1} \in G^{(s)}(\Omega)$, where $\Omega = x(U)$. This definition generalizes for vector and tensor fields.

Remark 8. The reason to treat X and Y differently in definitions of $G^{(s)}(X \times Y)$ and $G^{m,(s)}(X \times Y)$ is that, in the theorems of section A.2, we need to distinguish between the regularity with respect to the space-time X and the regularity with respect to the parametrization of the initial data.

Remark 9. We could similarly define for manifolds the analog of the other Gevrey spaces introduce above. However, this can be somewhat cumbersome and not always natural. In particular, the spaces $G^{m,(s)}$ require a distinguished coordinate that plays the role of time. This can always be done locally, and it can be done for globally hyperbolic manifolds if we fix a particular foliation in terms of space-like slices (as done, e.g., in [10, 12]), although it is debatable how canonical this is. Here we prefer to avoid extra complications, i.e., we in fact only need the definition of $G^{(s)}(\Sigma)$, which is used for the construction of appropriate local coordinates and the construction of the initial data for the system in $\mathbb{R}^4$ (sections 3.2 and 3.3) and in the results of section 3.6. The bulk of the proofs are carried out for the system in $\mathbb{R}^4$, where all the different Gevrey spaces play a role. It follows that the solution in $\mathbb{R}^4$ is in particular smooth, giving rise to a smooth globally hyperbolic development. Note that for the conclusion of Theorem 2.2 it is not needed to assert that the full solution enjoys certain Gevrey regularity.

For more about Gevrey spaces, see, e.g., [28, 39]. We remark that the terminology "local" and the notation G_{loc} are not standard.

A.2. **The Cauchy problem.** Let $a = a(x, \partial^k)$, $x \in X$, be a linear differential operator of order k. We can write

$$a(x, \partial^k) = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index. Let $p(x, \partial^k)$ be the principal part of $a(x, \partial^k)$, i.e.,

$$p(x, \partial^k) = \sum_{|\alpha| = k} a_\alpha(x) \partial^\alpha.$$

At each point $x \in X$ and for each co-vector $\xi \in T^*_x X$, where $T^* X$ is the cotangent bundle of X, we can associate a polynomial of order k in the cotangent space $T^*_x X$ obtained by replacing the derivatives by $\xi \in T^*_x X$. More precisely, for each k^{th} order derivative in $a(x, \partial^k)$, i.e.,

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

$|\alpha| = k$, we associate the polynomial

$$\xi^\alpha \equiv \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n},$$

where $\xi = (\xi_0, \xi_1, \xi_2, \dots, \xi_n) \in T^*_x X$, forming in this way the polynomial

$$p(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.$$

Clearly, $p(x, \xi)$ is a homogeneous polynomial of degree k. It is called the characteristic polynomial (at x) of the operator a.

The cone $V_x(p)$ of p in $T^*_x X$ is defined by the equation

$$p(x, \xi) = 0.$$

Definition A.3. With the above notation, $p(x, \xi)$ is called a hyperbolic polynomial (at x) if there exists $\zeta \in T^*_x X$ such that every straight line through ζ that does not contain the origin intersects the cone $V_x(p)$ at k real distinct points. The differential operator $a(x, \partial^k)$ is called a hyperbolic operator (at x) if $p(x, \xi)$ is hyperbolic.

Leray proved in [26] that (if X is at least three-dimensional) if $p(x, \xi)$ is hyperbolic at x, then the set of points ζ satisfying the condition of Definition A.3 forms the interior of two opposite half-cones $\Gamma^{*,+}_x(a)$, $\Gamma^{*,-}_x(a)$, with $\Gamma^{*,\pm}_x(a)$ non-empty, with boundaries that belong to $V_x(p)$.

Remark 10. Another way of stating Definition A.3 is as follows. Given $\zeta \in T_x X$, consider a non-zero vector θ that is not parallel to ζ and form the line $\lambda \zeta + \theta$, where $\lambda \in \mathbb{R}$ is a parameter. We then require this line to intersect the cone $V_x(p)$ at k distinct real points. An equivalent definition of hyperbolic polynomials is as follows [9]: $p(x,\xi)$ is hyperbolic at x if for each non-zero $\xi = (\xi_0,\dots,\xi_n) \in T^*_x X$, the equation $p(x,\xi)=0$ has k distinct real roots $\xi_0 = \xi_0(\xi_1,\dots,\xi_n)$.

With applications to systems in mind, we next consider the $N\times N$ diagonal linear differential operator matrix

$$A(x,\partial) = \begin{pmatrix} a^1(x,\partial^{k_1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^N(x,\partial^{k_N}) \end{pmatrix}.$$

Each $a^J(x,\partial^{k_J})$, $J=1,\dots,N$ is a linear differential operator of order k_J.

Definition A.4. The operator $A(x,\partial)$ is called Leray-Ohya hyperbolic (at x) if:

(i) The characteristic polynomial $p^J(x,\xi)$ of each $a^J(x,\partial^{k_J})$ is a product of hyperbolic polynomials, i.e.

$$p^J(x,\xi) = p^{J,1}(x,\xi)\cdots p^{J,r_J}(x,\xi),\ J=1,\dots,N,$$

where each $p^{J,q}(x,\xi)$, $q=1,\dots,r_J$, $J=1,\dots,N$, is a hyperbolic polynomial.

(ii) The two opposite convex half-cones,

$$\Gamma^{*,+}_x(A) = \bigcap_{J=1}^{N}\bigcap_{q=1}^{r_J}\Gamma^{*,+}_x(a^{J,q}),\ \text{ and }\ \Gamma^{*,-}_x(A) = \bigcap_{J=1}^{N}\bigcap_{q=1}^{r_J}\Gamma^{*,-}_x(a^{J,q}),$$

have a non-empty interior. Here, $\Gamma^{*,\pm}_x(a^{J,q})$ are the half-cones associated with the hyperbolic polynomials $p^{J,q}(x,\xi)$, $q=1,\dots,r_J$, $J=1,\dots,N$.

Remark 11. When the above hyperbolicity properties hold for every x, we call the corresponding operators hyperbolic (we can also talk about hyperbolicity in an open set, a certain region, etc.). When we say that an operator is Leray-Ohya hyperbolic on the whole space (or in an open set, etc.), this means not only that Definition A.4 applies for every x, but also that the numbers r_J and the degree of the polynomials $p^{J,q}(x,\xi)$, $q=1,\dots,r_J$, $J=1,\dots,N$, do not change with x.

Definition A.5. We define the dual convex half-cone $C^+_x(A)$ at T_xX as the set of $v\in T_xX$ such that $\xi(v)\geq 0$ for every $\xi\in\Gamma^{*,+}_x(A)$; $C^-_x(A)$ is analogously defined, and we set $C_x(A) = C^+_x(A)\cup C^-_x(A)$. If the convex cones $C^+_x(A)$ and $C^-_x(A)$ can be continuously distinguished with respect to $x\in X$, then X is called time-oriented (with respect to the hyperbolic form provided by the operator A). A path in X is called future (past) time-like with respect to A if its tangent at each point $x\in X$ belongs to $C^+_x(A)$ ($C^-_x(A)$), and future (past) causal if its tangent at each point $x\in X$ belongs or is tangent to $C^+_x(A)$ ($C^-_x(A)$). A regular surface Σ is called space-like with respect to A if $T_x\Sigma$ ($\subset T_xX$) is exterior to $C_x(A)$ for each $x\in\Sigma$. It follows that for a time-oriented X, the concepts of causal past, future, domains of dependence and influence of a set can be defined in the same way one does when the manifold is endowed with a Lorentzian metric. We refer the reader to [26] for details. Here we need only the following: the causal past $J^-(x)$ of a point $x\in X$ is the set of points that can be joined to x by a past causal curve.

Remark 12. The definitions in Definition A.5 endow X with a causal structure provided by the operator A. Despite the similar terminology, however, it should be noticed that all of the above definitions depend only on the structure of the operator A, and do not require an a priori Lorentzian metric on X. The case of interest in general relativity, however, is when the causal structure of the space-time is connected with that of A. In this regard, the following observation is useful. Suppose that X has a Lorentzian metric g. For causal solutions of the systems of equations here described (see Theorem A.11 below) to be causal in the sense of general relativity, one needs that, for all $x \in X$, $C_x^{\pm}(A) \subseteq K_x^{\pm}$, where $K_x^{\pm}$ are the two halves of the light-cone $\{g_{\mu\nu}\xi^\mu\xi^\nu \leq 0\}$. By duality, this is equivalent to saying that in the cotangent spaces we have $K_x^{*,\pm} \subseteq \Gamma_x^{*,+}(A)$, where $K_x^{*,\pm}$ are the two halves of the dual light-cone $\{g^{\mu\nu}\xi_\mu\xi_\nu \leq 0\}$.

Next, we consider the following quasi-linear system of differential equations

$$A(x, U, \partial)U = B(x, U), \tag{19}$$

where $A(x, U, \partial)$ is the $N \times N$ diagonal matrix

$$A(x, U, \partial) = \begin{pmatrix} a^1(x, U, \partial^{k_1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^N(x, U, \partial^{k_N}) \end{pmatrix},$$

with $a^J(x, U, \partial^{k_J})$, $J = 1, \ldots, N$ differential operators of order k_J. $B(x, U)$ is the vector

$$B(x, U) = (b^J(x, U)),\ J = 1, \ldots, N,$$

and the vector

$$U(x) = (U^I(x)),\ I = 1, \ldots, N$$

is the unknown. Notice that because a_J is allowed to depend on U, the above system is in general non-linear.

Definition A.6. The system $A(x, U, \partial)U = B(x, U)$ is called a Leray system if it is possible to attach to each unknown u^I an integer $m_I \geq 0$, and to each equation J of the system an integer $n_J \geq 0$, such that:

(i) $k_J = m_J - n_J$, $J = 1, \ldots, N$;

(ii) the functions b^J and the coefficients of the differential operators a^J are[9] functions of x, of u^I, and of the derivatives of u^I of order at most $m_I - n_J - 1$, $I, J = 1 \ldots, N$. If for some I and some J, $m_I - n_J < 0$, then the corresponding a^J and b^J do not depend on u^I.

Remark 13. The indices m_I and n_J in Definition A.6 are defined up to an additive integer.

Definition A.7. A Leray-Ohya system (with diagonal principal part) is a Leray system where the matrix A is Leray-Ohya hyperbolic. In the quasi-linear case, since the operators a depend on U, we need to specify a function U that is plugged into $A(x, U, \partial)$ in order to compute the characteristic polynomials. In this case we talk about a Leray-Ohya system for the function U. The primary case of interest is when U assumes the values of the given Cauchy data.

When considering a quasi-linear system, we write $p(x, U, \xi)$ and similar expressions to indicate the dependence on U.

We now formulate the Cauchy problem for Leray systems.

[9]The regularity required for the coefficients a^J and b^J depends on particular applications and context. For instance, for Theorem A.10 the required regularity is specified. Similarly, in Definition A.8, one needs to take derivatives of these quantities up to order n_J, thus they need to be at least as many times differentiable.

Definition A.8. Let Σ be a regular hypersurface in X, which we assume for simplicity to be given by $\{x^0 = 0\}$. The Cauchy data on Σ for a Leray system in X consists of the values of $U = (u^I)$ and their derivatives up to order $m_I - 1$ on Σ, i.e., $\partial^\alpha u^I\big|_\Sigma$, $|\alpha| \leq m_I - 1$, $I = 1, \ldots, N$. The Cauchy data is required to satisfy the following compatibility conditions. If $V = (v^I)$ is an extension of the Cauchy data defined in a neighborhood of Σ, i.e. $\partial^\alpha v^I\big|_\Sigma = \partial^\alpha u^I\big|_\Sigma$, $|\alpha| \leq m_I - 1$, $I = 1, \ldots, N$, then the difference $a^J(x, V, \partial)U - b^J(x, V)$ and its derivatives of order less than n_J vanish on Σ, for $J = 1, \ldots, N$. When to a Leray system $A(x, U, \partial)U = B(x, U)$ we prescribe initial data satisfying these conditions, we say that we have a Cauchy problem for $A(x, U, \partial)U = B(x, U)$.

Notice that by definition, the Cauchy data for a Leray system satisfies the aforementioned compatibility conditions. We also introduce the following notions related to the Cauchy problem for a Leray system.

Assumption 2. Consider the Cauchy problem for a Leray system $A(x, U, \partial)U = B(x, U)$. Let Y be an open set of $\mathbb{R}^L$, where L equals the number of derivatives of u^J of order less or equal to $\max_I m_I - n_J$, $J = 1, \ldots, N$, and such that Y contains the closure of the values taken by the Cauchy data on Σ. It is convenient to consider $A(x, U, \partial)$ as a differential operator defined over $X \times Y$, as follows. We shall assume that there exists a differential operator $\widetilde{A}(x, y, \partial)$ defined over $X \times Y$ with the following property. If $(x, y) \in X \times Y$ and $V = (v^J)$ is a sufficiently regular function on Σ such that $y = (\partial^{\max_I m_I - n_J} v^J(x))_{J=1,\ldots,N}$, then $A(x, V(x), \partial) = \widetilde{A}(x, y, \partial)$. We shall write $A(x, y, \partial)$ for $\widetilde{A}(x, y, \partial)$.

Definition A.9. Consider the Cauchy problem for a Leray system $A(x, U, \partial)U = B(x, U)$. Let Σ and Y be as in Definition A.8 and Assumption 2, respectively. Denote by $\mathcal{A}^s(\Sigma, I)$ the set of $V = (v^J) \in G^{(s)}(\Sigma)$, $J = 1, \ldots, N$, such that $(\partial^{\max_I m_I - n_J} v^J(x))_{J=1,\ldots,N} \in Y$ for all $x \in \Sigma$.

We are now ready to state the results of this appendix. We use the above notation and definitions in the statement of the theorems below.

Theorem A.10 (Existence and uniqueness)**.** *Consider the Cauchy problem for (19). Suppose that the Cauchy data is in $G^{(s)}(\Sigma)$, and that*

$$a^J(\cdot, \cdot, \partial^{k_J}) \in G^{n_J,(s)}_{loc}(X \times Y), \ \ \textit{and} \ \ b^J(\cdot, \cdot) \in G^{n_J,(s)}(X \times Y).$$

Suppose that for any $V \in \mathcal{A}^s(\Sigma, Y)$ the system is Leray-Ohya hyperbolic with indices m_I and n_J; thus for all $x \in \Sigma$, each $p^J(x, V, \xi)$ is the product of r_J hyperbolic polynomials,

$$p^J(x, V, \xi) = p^{J,1}(x, V, \xi) \cdots p^{J,r_J}(x, V, \xi), \; J = 1, \ldots, N.$$

Suppose that each $p^{J,q+1}(x, V, \xi)$, $q = 0, \ldots, r_J - 1$, depends on at most $m_I - m_{J,q} - r_I + q$ derivatives of v^I, $I = 1, \ldots, N$, where

$$m_{J,q} = n_J + \deg(p^{J,1}) + \cdots + \deg(p^{J,q}), \; m_{J,r_J} = m_J, \; m_{J,0} = n_J.$$

Above, $\deg(p^{J,q})$ *is the degree, in ξ, of the polynomial $p^{J,q}(x, V, \xi)$.*

Denote by $a^J_{q+1}(x, y, \partial)$ the differential operator associated with $p^{J,q+1}$. Assume that

$$a^J_{q+1}(\cdot, \cdot, \partial) \in G^{m_{J,q} - q,(s)}_{loc}(X \times Y).$$

Let $0 \leq g_I \leq r_I$ be the smallest integers such that $a^J(x, V, \partial^{m_J - n_J})$ and $b^J(x, V)$ depend on at most $m_I - n_J - r_I + g_I$ derivatives of v^I, $I = 1, \dots, N$, $J = 1, \dots, N$. Finally, assume that

$$1 \leq s \leq \frac{r_J}{g_J} \text{ and } \frac{n}{2} + r^J < n_J,\ J = 1, \dots, N.$$

Then, there exists a $T' > 0$ and a solution $U = (u^I)$ to the Cauchy problem for (19) and defined on $[0, T') \times \mathbb{R}^n \subseteq X$. The solution satisfies

$$u^I \in G^{m_I,(s)}([0, T') \times \mathbb{R}^n),\ I = 1, \dots, N.$$

Furthermore, the solution is unique in this regularity class.

Theorem A.11 (Causality)**.** *Assume the same hypotheses of Theorem A.10, and suppose further that*

$$1 \leq s < \frac{r_J}{g_J},\ J = 1, \dots, N.$$

Let T' and U be as in the conclusion of Theorem A.10. Then, if T' is sufficiently small, the operator $A(x, U, \partial)$ is Leray-Ohya hyperbolic (thus the causal past of a point is well-defined), and for each $x \in [0, T') \times \mathbb{R}^n$, $U(x)$ depends only on $U_0|_{J^-(x) \cap \Sigma}$, where U_0 is the Cauchy data.

Remark 14. Theorem A.10 assumes that the system is Leray-Ohya hyperbolic for $V \in \mathcal{A}(\Sigma, Y)$, which is essentially the space of values near the initial data. (Naturally, it would not make sense to require the system to be Leray-Ohya hyperbolic for the yet to be proven to exist solution U.) Once U is constructed, one can then ask whether the system is Leray-Ohya hyperbolic for U. This will be the case if T' is small, since in this case the values of U will be close to those of the initial data by continuity, guaranteeing that $U(x) \in \mathcal{A}(\Sigma, Y)$.

Theorems A.10 and A.11 are proven in [28] (reprinted in [29]).

We now consider a system whose principal part is not necessarily diagonal. The definition of a Leray system depends only on the existence of the indices m_I and n_J with the stated properties, and thus can be extended to non-diagonal systems.

Definition A.12. Consider a system of N partial differential equations and N unknowns in X, and denote the unknown as $U = (u^I)$, $I = 1, \dots, N$. The system is a (not necessarily diagonal in the principal part) Leray system if it is possible to attach to each unknown u^I a non-negative integer m_I and to each equation a non-negative integer n_J, such that the system reads

$$h^J_I(x, \partial^{m_K - n_J - 1} u^K, \partial^{m_I - n_J}) u^I + b^J(x, \partial^{m_K - n_J - 1} u^K) = 0,\ J = 1, \dots, N. \tag{20}$$

Here, $h^J_I(x, \partial^{m_K - n_J - 1} u^K, \partial^{m_I - n_J})$ is a homogeneous differential operator of order $m_I - n_J$ (which can be zero), whose coefficients depend on at most $m_K - n_J - 1$ derivatives of u^K, $K = 1, \dots N$, and there is a sum over I in $h^J_I(\cdot) u^I$. The remaining terms, $b^J(x, \partial^{m_K - n_J - 1} u^K)$, also depend on at most $m_K - n_J - 1$ derivatives of u^K, $K = 1, \dots N$. As before, these indices are defined only up to an overall additive integer.

As done above, for a given sufficiently regular U, $h^J_I(x, \partial^{m_K - n_J - 1} U^K, \partial^{m_I - n_J})$ are well-defined linear operators, and we can ask about their hyperbolicity properties. The case of interest will be, again, when we evaluate these operators at some given Cauchy data.

Write (20) in matrix form as

$$H(x,U,\partial)U = B(x,U). \tag{21}$$

Definition A.13. The characteristic determinant of (21) at $x \in X$ and for a given U is the polynomial $p(x,\xi)$ in the co-tangent space T_x^*X, $\xi \in T_x^*X$, given by

$$p(x,U,\xi) = \det(H(x,U,\xi)). \tag{22}$$

Note that p is a homogeneous polynomial of degree

$$\ell \equiv \sum_{I=1}^{N} m_I - \sum_{J=1}^{N} n_J.$$

Under appropriate conditions, (21) can be transformed into a Leray-Ohya system of the form (19), i.e., with diagonal principal part. More precisely, we have the following.

Theorem A.14 (Diagonalization)**.** *Consider (21). Suppose that the characteristic determinant (22) at a given U is not identically zero, and it is the product of Q hyperbolic polynomials, i.e.,*

$$p(x,U,\xi) = p_1(x,U,\xi)\cdots p_Q(x,U,\xi).$$

Let d_q be the degree of $p_q(x,U,\xi)$, $q=1,\ldots,Q$, and suppose that

$$\max_q d_q \geq \max_I m_I - \min_J n_J.$$

Finally, assume that

$$\ell \geq \max_I m_I - \min_J n_J.$$

Then, there exists a $N\times N$ matrix $C(x,U,\partial)$ of differential operators whose coefficients depend on U, such that

$$C(x,U,\partial)H(x,U,\partial)U = \mathbb{I}\, p(x,U,\partial)U + \widetilde{B}_1(x,U),$$

and

$$C(x,U,\partial)B(x,U) = \widetilde{B}_2(x,U),$$

where $\mathbb{I}$ is the $N\times N$ identity matrix, $p(x,U,\partial)$ is the differential operator associated with $p(x,U,\xi)$, and $\widetilde{B}_1(x,U)$ and $\widetilde{B}_2(x,U)$ depend on at most $\ell-1$ derivatives of U, as do the coefficients of the operator $p(x,U,\xi)$. Furthermore, there is a choice of indices that makes the system

$$\mathbb{I}\, p(x,U,\partial)U = \widetilde{B}_2(x,U) - \widetilde{B}_1(x,U) \tag{23}$$

into a Leray system. In particular, if the intersections $\cap_q \Gamma_x^{*,+}(a^q)$ and $\cap_q \Gamma_x^{*,-}(a^q)$, where $\Gamma_x^{*,\pm}(a^q)$ are the half-cones associated with the hyperbolic polynomials $p_q(x,U,\xi)$, have non-empty interiors, then (23) is a Leray-Ohya system with diagonal principal part in the sense of definition A.7.

Theorem A.14 is proven in [6].

Definition A.15. Under the hypotheses of Theorem A.14, the number $\frac{Q}{Q-1}$ is called the Gevrey index of the system.

Remark 15. Suppose that (23) forms a Leray-Ohya system in the sense of definition A.7, i.e., the half-cones have non-empty interiors as stated in Theorem A.14. It can then be shown [6] that a value of s sufficient to apply Theorems A.10 and A.11 is $1 \leq s < \frac{Q}{Q-1}$.

Let us make a brief comment about the proofs of the above results. Theorem A.10 is proven as follows. First, one solves the associated linear problem. This is done by a method of majorants reminiscent of the Cauchy-Kowalevskaya theorem. One uses the fact that Gevrey functions admit a formal series expansion that provides a consistent way of constructing successive approximating solutions to the problem. The non-linear problem is then treated via a fixed point argument, upon solving successive linear problems. Theorem A.11 is obtained by a Holmgren type of argument. We remark that the assumption that $p^{J,q+1}(x, V, \xi)$, $q = 0, \dots, r_J - 1$, depends on at most $m_I - m_{J,q} - r_I + q$ derivatives of v^I, $I = 1, \dots, N$, ensures that the coefficients of the associated differential operators $a^{J,q+1}(x, U, \partial)$ do not depend on too many derivatives of U, as it should be in the treatment of quasi-linear equations.

Theorem A.14 is based on the following identity:

$$c^T a = \det(a), \tag{24}$$

where a is an $N \times N$ invertible matrix and c^T the transpose of the co-factor matrix. At the level of differential operators, this identity produces the lower order terms $\widetilde{B}_1$. One then needs to match the order of the resulting differential operators and lower order terms with appropriate indices satisfying the definition of a Leray system. This is possible under the conditions on d_q and ℓ stated in the theorem.

Appendix B. **Derivation of the equations of motion.** In this section we give the derivation of (6) and (7). The derivation of (6) is standard and we include it here for the reader's convenience, thus let us start with (6). Let

$${}^{(0)}t_{\alpha\beta} = \frac{4}{3} u_\alpha u_\beta \epsilon + \frac{1}{3} g_{\alpha\beta}\epsilon, \tag{25}$$

and denote the third to ninth terms in (1) by ${}^{(1)}t_{\alpha\beta}$ to ${}^{(7)}t_{\alpha\beta}$, respectively. Explicitly,

$$\begin{aligned}
{}^{(1)}t_{\alpha\beta} &= -\eta \pi_\alpha^\mu \pi_\beta^\nu (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} g_{\mu\nu} \nabla_\lambda u^\lambda),\\
{}^{(2)}t_{\alpha\beta} &= \lambda (u_\alpha u^\mu \nabla_\mu u_\beta + u_\beta u^\mu \nabla_\mu u_\alpha),\\
{}^{(3)}t_{\alpha\beta} &= \frac{1}{3} \chi \pi_{\alpha\beta} \nabla_\mu u^\mu,\\
{}^{(4)}t_{\alpha\beta} &= \chi u_\alpha u_\beta \nabla_\mu u^\mu,\\
{}^{(5)}t_{\alpha\beta} &= \frac{\lambda}{4\epsilon} (u_\alpha \pi_\beta^\mu \nabla_\mu \epsilon + u_\beta \pi_\alpha^\mu \nabla_\mu \epsilon),\\
{}^{(6)}t_{\alpha\beta} &= \frac{3\chi}{4\epsilon} u_\alpha u_\beta u^\mu \nabla_\mu \epsilon,\\
{}^{(7)}t_{\alpha\beta} &= \frac{\chi}{4\epsilon} \pi_{\alpha\beta} u^\mu \nabla_\mu \epsilon,
\end{aligned}$$

so that

$$T_{\alpha\beta} = {}^{(0)}t_{\alpha\beta} + {}^{(1)}t_{\alpha\beta} + \cdots {}^{(7)}t_{\alpha\beta}.$$

B.1. **Calculation of $\nabla_\alpha {}^{(1)}t^\alpha_\beta$.** We have

$$\begin{aligned}\nabla_\alpha {}^{(1)}t^\alpha_\beta &= -\eta\pi^{\alpha\mu}\pi^\nu_\beta(\nabla_\alpha\nabla_\mu u_\nu + \nabla_\alpha\nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\alpha\nabla_\lambda u^\lambda) \\ &\quad + \nabla_\alpha(\eta\pi^{\alpha\mu}\pi^\nu_\beta)(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda).\end{aligned} \tag{26}$$

Compute

$$\begin{aligned}\pi^\nu_\beta\nabla_\alpha\nabla_\mu u_\nu &= (g^\nu_\beta + u_\beta u^\nu)\nabla_\alpha\nabla_\mu u_\nu = \nabla_\alpha\nabla_\mu u_\beta + u_\beta u^\nu\nabla_\alpha\nabla_\mu u_\nu \\ &= \nabla_\alpha\nabla_\mu u_\beta + u_\beta\nabla_\alpha(u^\nu\nabla_\mu u_\nu) - u_\beta\nabla_\alpha u^\nu\nabla_\mu u_\nu \\ &= \nabla_\alpha\nabla_\mu u_\beta - u_\beta\nabla_\alpha u^\nu\nabla_\mu u_\nu,\end{aligned}$$

so that

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta\nabla_\alpha\nabla_\mu u_\nu &= -\eta\pi^{\alpha\mu}(\nabla_\alpha\nabla_\mu u_\beta - \nabla_\alpha u^\nu\nabla_\mu u_\nu) \\ &= -\eta(g^{\alpha\mu} + u^\alpha u^\mu)\nabla_\alpha\nabla_\mu u_\beta + \eta\pi^{\alpha\mu}\nabla_\alpha u^\nu\nabla_\mu u_\nu \\ &= -\eta g^{\alpha\mu}\nabla_\alpha\nabla_\mu u_\beta + u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \eta\pi^{\alpha\mu}\nabla_\alpha u^\nu\nabla_\mu u_\nu.\end{aligned} \tag{27}$$

Similarly, we find

$$\begin{aligned}\pi^{\alpha\mu}\nabla_\alpha\nabla_\nu u_\mu &= (g^{\alpha\mu} + u^\alpha u^\mu)\nabla_\alpha\nabla_\nu u_\mu = g^{\alpha\mu}\nabla_\alpha\nabla_\nu u_\mu + u^\alpha u^\mu\nabla_\alpha\nabla_\nu u_\mu \\ &= \nabla_\alpha\nabla_\nu u^\alpha - u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu,\end{aligned}$$

so that

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta\nabla_\alpha\nabla_\nu u_\mu &= -\eta\pi^\nu_\beta(\nabla_\alpha\nabla_\nu u^\alpha - u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu) \\ &= -\eta g^\nu_\beta\nabla_\alpha\nabla_\nu u^\alpha - \eta u_\beta u^\nu\nabla_\alpha\nabla_\nu u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu.\end{aligned} \tag{28}$$

But

$$\nabla_\alpha\nabla_\nu u^\alpha = \nabla_\nu\nabla_\alpha u^\alpha + R_{\nu\alpha}u^\alpha,$$

so that (28) becomes

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta\nabla_\alpha\nabla_\nu u_\mu &= -\eta g^\nu_\beta(\nabla_\nu\nabla_\alpha u^\alpha + R_{\nu\alpha}u^\alpha) - \eta u_\beta u^\nu(\nabla_\nu\nabla_\alpha u^\alpha + R_{\nu\alpha}u^\alpha) \\ &\quad + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu \\ &= -\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha - \eta g^\nu_\beta R_{\nu\alpha}u^\alpha - \eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha \\ &\quad - \eta u_\beta u^\nu R_{\nu\alpha}u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu.\end{aligned} \tag{29}$$

Next compute

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta(-\frac{2}{3}g_{\mu\nu}\nabla_\alpha\nabla_\lambda u^\lambda) &= \frac{2}{3}\eta\pi^{\alpha\mu}\pi_{\beta\mu}\nabla_\alpha\nabla_\lambda u^\lambda \\ &= \frac{2}{3}\eta\pi^\alpha_\beta\nabla_\alpha\nabla_\lambda u^\lambda = \frac{2}{3}\eta(g^\alpha_\beta + u^\alpha u_\beta)\nabla_\alpha\nabla_\lambda u^\lambda \\ &= \frac{2}{3}\eta g^\alpha_\beta\nabla_\alpha\nabla_\lambda u^\lambda + \frac{2}{3}\eta u_\beta u^\alpha\nabla_\alpha\nabla_\lambda u^\lambda.\end{aligned} \tag{30}$$

Plugging (27), (29), and (30) into (26) we find

$$\begin{aligned}\nabla_\alpha {}^{(1)}t^\alpha_\beta &= -\eta g^{\alpha\mu}\nabla_\alpha\nabla_\mu u_\beta - \eta u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \eta u_\beta\pi^{\alpha\mu}\nabla_\alpha u_\nu\nabla_\mu u^\nu - \eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha \\ &\quad - \eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha - \eta R_{\beta\alpha}u^\alpha - \eta u_\beta R_{\nu\alpha}u^\nu u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu \\ &\quad + \frac{2}{3}\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha + \frac{2}{3}\eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha \\ &\quad + \nabla_\alpha(\eta\pi^{\alpha\mu}\pi^\nu_\beta)(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda).\end{aligned}$$

We now group the first two terms, the fourth term with the ninth term, and the fifth term with the tenth term, to find

$$\begin{aligned}\nabla_\alpha{}^{(1)}t^\alpha_\beta = &-\eta(g^{\alpha\mu}+u^\alpha u^\beta)\nabla_\alpha\nabla_\mu u_\beta - \frac{1}{3}\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha \\ &- \frac{1}{3}\eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha + {}^{(1)}B_\beta,\end{aligned} \tag{31}$$

where

$$\begin{aligned}{}^{(1)}B_\beta = &\ \eta u_\beta\pi^{\alpha\mu}\nabla_\alpha u_\nu\nabla_\mu u^\nu - \eta R_{\beta\alpha}u^\alpha - \eta u_\beta R_{\nu\alpha}u^\nu u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu \\ &+ \nabla_\alpha(\eta\pi^{\alpha\mu}\pi^\nu_\beta)(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda).\end{aligned} \tag{32}$$

B.2. **Calculation of $\nabla_\alpha{}^{(2)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha{}^{(2)}t^\alpha_\beta &= \nabla_\alpha\big[\lambda(u^\alpha u^\mu\nabla_\mu u_\beta + u_\beta u^\mu\nabla_\mu u^\alpha)\big] \\ &= \lambda(u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + u_\beta u^\mu\nabla_\alpha\nabla_\mu u^\alpha) + \nabla_\alpha(\lambda u^\alpha u^\mu)\nabla_\mu u_\beta \\ &+ \nabla_\alpha(\lambda u_\beta u^\mu)\nabla_\mu u^\alpha.\end{aligned}$$

Using $\nabla_\alpha\nabla_\mu u^\alpha = \nabla_\mu\nabla_\alpha u^\alpha + R_{\mu\alpha}u^\alpha$ we find

$$\nabla_\alpha{}^{(2)}t^\alpha_\beta = \lambda u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \lambda u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha + {}^{(2)}B_\beta, \tag{33}$$

where

$${}^{(2)}B_\beta = \lambda u_\beta R_{\mu\alpha}u^\mu u^\alpha + \nabla_\alpha(\lambda u^\alpha u^\mu)\nabla_\mu u_\beta + \nabla_\alpha(\lambda u_\beta u^\mu)\nabla_\mu u^\alpha. \tag{34}$$

B.3. **Calculation of $\nabla_\alpha{}^{(3)}t^\alpha_\beta$.** Compute

$$\nabla_\alpha{}^{(3)}t^\alpha_\beta = \nabla_\alpha\big(\frac{1}{3}\pi^\alpha_\beta\nabla_\mu u^\mu\big) = \frac{1}{3}\chi\pi^\alpha_\beta\nabla_\alpha\nabla_\mu u^\mu + \frac{1}{3}\nabla_\alpha(\chi\pi^\alpha_\beta)\nabla_\mu u^\mu,$$

so that

$$\nabla_\alpha{}^{(3)}t^\alpha_\beta = \chi\frac{1}{3}g^\mu_\beta\nabla_\mu\nabla_\alpha u^\alpha + \frac{1}{3}\chi u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha + {}^{(3)}B_\beta, \tag{35}$$

where

$${}^{(3)}B_\beta = \frac{1}{3}\nabla_\alpha(\chi\pi^\alpha_\beta)\nabla_\mu u^\mu. \tag{36}$$

B.4. **Calculation of $\nabla_\alpha{}^{(4)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha{}^{(4)}t^\alpha_\beta &= \nabla_\alpha\big(\chi u^\alpha u_\beta\nabla_\mu u^\mu\big) \\ &= \chi u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha + {}^{(4)}B_\beta,\end{aligned} \tag{37}$$

where

$${}^{(4)}B_\beta = \nabla_\alpha(\chi u^\alpha u_\beta)\nabla_\mu u^\mu. \tag{38}$$

B.5. **Calculation of $\nabla_\alpha {}^{(5)}t^\alpha_\beta$.** Compute

$$
\begin{aligned}
\nabla_\alpha {}^{(5)}t^\alpha_\beta &= \nabla_\alpha\Big[\frac{\lambda}{4\epsilon}(u^\alpha \pi^\mu_\beta \nabla_\mu \epsilon + u_\beta \pi^{\alpha\mu}\nabla_\mu\epsilon\Big] \\
&= \frac{\lambda}{4\epsilon}u^\alpha \pi^\mu_\beta \nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta\pi^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\big[\frac{\lambda}{4\epsilon}(u^\alpha\pi^\mu_\beta + u_\beta\pi^{\alpha\mu})\big]\nabla_\mu\epsilon \\
&= \frac{\lambda}{4\epsilon}u^\alpha g^\mu_\beta \nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta g^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon \\
&+ \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\big[\frac{\lambda}{4\epsilon}(u^\alpha\pi^\mu_\beta + u_\beta\pi^{\alpha\mu})\big]\nabla_\mu\epsilon.
\end{aligned}
$$

We rearrange the terms, swapping the first and third terms, so that

$$
\begin{aligned}
\nabla_\alpha {}^{(5)}t^\alpha_\beta &= \frac{\lambda}{4\epsilon}u_\beta g^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon \\
&+ \frac{\lambda}{4\epsilon}u^\alpha g^\mu_\beta\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + {}^{(5)}B_\beta,
\end{aligned} \tag{39}
$$

where

$$
{}^{(5)}B_\beta = \nabla_\alpha\big[\frac{\lambda}{4\epsilon}(u^\alpha\pi^\mu_\beta + u_\beta\pi^{\alpha\mu})\big]\nabla_\mu\epsilon. \tag{40}
$$

B.6. **Calculation of $\nabla_\alpha {}^{(6)}t^\alpha_\beta$.** Compute

$$
\begin{aligned}
\nabla_\alpha {}^{(6)}t^\alpha_\beta &= \nabla_\alpha\Big[\frac{3\chi}{4\epsilon}u^\alpha u_\beta u^\mu\nabla_\mu\epsilon\Big] = \frac{3\chi}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\Big[\frac{3\chi}{4\epsilon}u^\alpha u_\beta u^\mu\Big]\nabla_\mu\epsilon \\
&= \frac{3\chi}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + {}^{(6)}B_\beta,
\end{aligned} \tag{41}
$$

where

$$
{}^{(6)}B_\beta = \nabla_\alpha\Big[\frac{3\chi}{4\epsilon}u^\alpha u_\beta u^\mu\Big]\nabla_\mu\epsilon. \tag{42}
$$

B.7. **Calculation of $\nabla_\alpha {}^{(7)}t^\alpha_\beta$.** Compute

$$
\begin{aligned}
\nabla_\alpha {}^{(7)}t^\alpha_\beta &= \nabla_\alpha\Big[\frac{\chi}{4\epsilon}\pi^\alpha_\beta u^\mu\nabla_\mu\epsilon\Big] = \frac{\chi}{4\epsilon}(g^\alpha_\beta + u^\alpha u_\beta)u^\mu\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\Big[\frac{\chi}{4\epsilon}\pi^\alpha_\beta u^\mu\Big]\nabla_\mu\epsilon \\
&= \frac{\chi}{4\epsilon}g^\alpha_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{\chi}{4\epsilon}u^\alpha u_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + {}^{(7)}B_\beta,
\end{aligned} \tag{43}
$$

where

$$
{}^{(7)}B_\beta = \nabla_\alpha\Big[\frac{\chi}{4\epsilon}\pi^\alpha_\beta u^\mu\Big]\nabla_\mu\epsilon. \tag{44}
$$

B.8. **Calculation of $\nabla_\alpha T^\alpha_\beta$.** Using (1), (25), (31), (33), (35), (37), (39), (41), and (43), we find

$$
\begin{aligned}
\nabla_\alpha T^\alpha_\beta &= -\eta(g^{\alpha\mu} + u^\alpha u^\beta)\nabla_\alpha\nabla_\mu u_\beta - \frac{1}{3}\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha - \frac{1}{3}\eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha \\
&+ \lambda u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \lambda u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha \\
&+ \chi\frac{1}{3}g^\mu_\beta\nabla_\mu\nabla_\alpha u^\alpha + \frac{1}{3}\chi u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha \\
&+ \chi u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha + \frac{\lambda}{4\epsilon}u_\beta g^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon \\
&+ \frac{\lambda}{4\epsilon}u^\alpha g^\mu_\beta\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{3\chi}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon \\
&+ \frac{\chi}{4\epsilon}g^\alpha_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{\chi}{4\epsilon}u^\alpha u_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + B_\beta,
\end{aligned} \tag{45}
$$

where the first three terms on the RHS of (45) come from (31), the fourth and fifth from (33), the sixth and seventh from (35), the eighth from (37), the ninth to twelfth from (39), the thirteenth from (41), the fourteenth and fifteenth from (43), and B_β is given by

$$B_\beta = {}^{(1)}B_\beta + {}^{(2)}B_\beta + {}^{(3)}B_\beta + {}^{(4)}B_\beta + {}^{(5)}B_\beta + {}^{(6)}B_\beta + {}^{(7)}B_\beta + \nabla_\alpha {}^{(0)}t^\alpha_\beta, \tag{46}$$

with ${}^{(1)}B_\beta, \ldots, {}^{(7)}B_\beta$ given by (32), (34), (36), (38), (40), (42), and (44), respectively, and ${}^{(0)}t^\alpha_\beta$ is given by (25). We now group the terms on the RHS of (45) as follows: the first and the fourth terms, the fifth and the eighth terms, the second and the sixth terms, the third and the seventh terms, the ninth, tenth, and thirteenth terms, the eleventh and fourteenth terms, and the twelfth and fifteenth terms. We obtain:

$$\begin{aligned}\nabla_\alpha T^\alpha_\beta &= (-\eta g^{\alpha\mu} + (\lambda - \eta) u^\alpha u^\mu) \nabla_\alpha \nabla_\mu u_\beta + (\lambda + \chi) u_\beta u^\mu \nabla_\mu \nabla_\alpha u^\alpha \\ &\quad + \frac{1}{3}(-\eta + \chi) g^\mu_\beta \nabla_\mu \nabla_\alpha u^\alpha + \frac{1}{3}(-\eta + \chi) u_\beta u^\mu \nabla_\mu \nabla_\alpha u^\alpha \\ &\quad + \frac{1}{4\epsilon} u_\beta (\lambda g^{\alpha\mu} + (\lambda + 3\chi) u^\alpha u^\mu) \nabla_\alpha \nabla_\mu \epsilon + \frac{1}{4\epsilon}(\lambda + \chi) g^\mu_\beta u^\alpha \nabla_\alpha \nabla_\mu \epsilon \\ &\quad + \frac{1}{4\epsilon}(\lambda + \chi) u_\beta u^\alpha u^\mu \nabla_\alpha \nabla_\mu \epsilon + B_\beta,\end{aligned} \tag{47}$$

where the first term on the RHS of (47) comes from the first and the fourth terms on the RHS of (45), the second term on the RHS of (47) comes from the fifth and the eighth terms on the RHS of (45), the third term on the RHS of (47) comes from second and the sixth terms on the RHS of (45), the fourth term on the RHS of (47) comes from the third and the seventh terms on the RHS of (45), the fifth term on the RHS of (47) comes from the ninth, tenth, and thirteenth terms on the RHS of (45), the sixth term on the RHS of (47) comes from the eleventh and fourteenth terms on the RHS of (45), the seventh term on the RHS of (47) comes from and the twelfth and fifteenth terms on the RHS of (45), and we used that $\nabla_\alpha \nabla_\mu \epsilon = \nabla_\mu \nabla_\alpha \epsilon$.

Expanding the covariant derivatives and using Notation 1 gives (7).

B.9. **Derivation of (6).** Let us first write (3) in trace reversed form. Tracing (3) gives

$$R = 4\Lambda - T,$$

where $T = g^{\alpha\beta} T_{\alpha\beta}$. (For (1) we in fact have $T = 0$, as it must be for a conformal tensor. But at this point we are writing Einstein's equations for a general tensor.) Plugging this for R in (3) gives

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} + \Lambda g_{\alpha\beta}.$$

We now proceed to compute $R_{\alpha\beta}$ in local coordinates. In coordinates, we have

$$R_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu} \Gamma^\mu_{\beta\lambda}.$$

Using the definition of the Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ gives

$$\begin{aligned}R_{\alpha\beta} &= -\frac{1}{2} g^{\mu\nu} \partial^2_{\mu\nu} g_{\alpha\beta} + \frac{1}{2}(g_{\alpha\lambda} \partial_\beta \Gamma^\lambda + g_{\beta\lambda} \partial_\alpha \Gamma^\lambda) \\ &\quad - \frac{1}{2}(\partial_\beta g^{\lambda\mu} \partial_\lambda g_{\alpha\mu} + \partial_\alpha g^{\lambda\mu} \partial_\lambda g_{\beta\mu}) - \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\beta\mu},\end{aligned}$$

where Γ^λ is given by

$$\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda_{\mu\nu}.$$

Using that in wave coordinates $\Gamma^\lambda = 0$ and recalling (1), the above gives (6).

Appendix C. **The characteristic determinant.** In this section we derive (11). Because of the structure of the system in (9) it suffices to compute the characteristic determinant of $m(U,\partial)$ in (10). Using Mathematica and (5) we find (we are not assuming $a_1 = 4$ at this point)

$$\det m(\widehat{U},\xi) = \widetilde{p}_1(\widehat{U},\xi)\widetilde{p}_2(\widehat{U},\xi)\widetilde{p}_3(\widehat{U},\xi),$$

where

$$\widetilde{p}_1(\widehat{U},\xi) = \frac{1}{12\widehat{\epsilon}}\eta^4(\widehat{u}^\mu\xi_\mu)^2,$$

$$\begin{aligned}\widetilde{p}_2(\widehat{U},\xi) = \big[&(a_2-1)(\widehat{u}^0)^2\xi_0^2 + (a_2-1)(\widehat{u}^1)^2\xi_1^2 - (\widehat{u}^2)^2\xi_2^2 + a_2(\widehat{u}^2)^2\xi_2^2 - 2\widehat{u}^2\widehat{u}^3\xi_2\xi_3\\ &+ 2a_2\widehat{u}^2\widehat{u}^3\xi_2\xi_3 - (\widehat{u}^3)^2\xi_3^2 + a_2(\widehat{u}^3)^2\xi_3^2 + \xi_0(2(-1+a_2)\xi_1\widehat{u}^0\widehat{u}^1\\ &+ 2(a_2-1)\xi_2\widehat{u}^0\widehat{u}^2 - 2\xi_3\widehat{u}^0\widehat{u}^3 + 2a_2\widehat{u}^0\widehat{u}^3\xi_3 - \xi^0)\\ &+\xi_1(2(-1+a_2)\widehat{u}^1\widehat{u}^2\xi_2 + 2(a_2-1)\widehat{u}^1\widehat{u}^3\xi_3 - \xi^1) - \xi_2\xi^2 - \xi_3\xi^3\big]^2\end{aligned}$$

and

$$\begin{aligned}\widetilde{p}_3(\widehat{U},\xi) = &-6(-2a_1\widehat{u}_0\widehat{u}^0 - a_2\widehat{u}_0\widehat{u}^0 + 2a_1a_2\widehat{u}_0\widehat{u}^0 + a_2^2\widehat{u}_0\widehat{u}^0 - 2a_1\widehat{u}_1\widehat{u}^1 - a_2\widehat{u}_1\widehat{u}^1\\ &+ 2a_1a_2\widehat{u}_1\widehat{u}^1 + a_2^2\widehat{u}_1\widehat{u}^1 - 2a_1\widehat{u}_2\widehat{u}^2 - a_2\widehat{u}_2\widehat{u}^2 + 2a_1a_2\widehat{u}_2\widehat{u}^2 + a_2^2\widehat{u}_2\widehat{u}^2\\ &- 2a_1\widehat{u}_3\widehat{u}^3 - a_2\widehat{u}_3\widehat{u}^3 + 2a_1a_2\widehat{u}_3\widehat{u}^3 + a_2^2\widehat{u}_3\widehat{u}^3)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1\\ &+ \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^4\\ &- 2(-a_2\widehat{u}_0 + 4a_1a_2\widehat{u}_0 + 3a_2^2\widehat{u}_0)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^0\\ &- 2(-a_2\widehat{u}_1 + 4a_1a_2\widehat{u}_1 + 3a_2^2\widehat{u}_1)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^1\\ &- 2(-a_2\widehat{u}_2 + 4a_1a_2\widehat{u}_2 + 3a_2^2\widehat{u}_2)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^2\\ &- 2(-a_2\widehat{u}_3 + 4a_1a_2\widehat{u}_3 + 3a_2^2\widehat{u}_3)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^3\\ &+ 5(3a_1\widehat{u}_0\widehat{u}^0 + 2a_2\widehat{u}_0\widehat{u}^0 + a_1a_2\widehat{u}_0\widehat{u}^0 + 3a_1\widehat{u}_1\widehat{u}^1 + 2a_2\widehat{u}_1\widehat{u}^1\\ &+ a_1a_2\widehat{u}_1\widehat{u}^1 + 3a_1\widehat{u}_2\widehat{u}^2 + 2a_2\widehat{u}_2\widehat{u}^2 + a_1a_2\widehat{u}_2\widehat{u}^2\\ &+ 3a_1\widehat{u}_3\widehat{u}^3 + 2a_2\widehat{u}_3\widehat{u}^3 + a_1a_2\widehat{u}_3\widehat{u}^3)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1\\ &+ \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^2(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3)\\ &+ (3a_1\widehat{u}_0 + 2a_2\widehat{u}_0 + a_1a_2\widehat{u}_0)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^0(\xi_0\xi^0\\ &+ \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) + (3a_1\widehat{u}_1 + 2a_2\widehat{u}_1 + a_1a_2\widehat{u}_1)(\xi_0\widehat{u}^0\\ &+ \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^1(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3)\\ &+ (3a_1\widehat{u}_2 + 2a_2\widehat{u}_2 + a_1a_2\widehat{u}_2)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1\\ &+ \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^2(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3)\\ &+ (3a_1\widehat{u}_3 + 2a_2\widehat{u}_3 + a_1a_2\widehat{u}_3)(\xi_0\widehat{u}^0\\ &+ \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^3(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3)\\ &+ (4a_2\widehat{u}_0\widehat{u}^0 - a_1a_2\widehat{u}_0\widehat{u}^0 + 4a_2\widehat{u}_1\widehat{u}^1 - a_1a_2\widehat{u}_1\widehat{u}^1 + 4a_2\widehat{u}_2\widehat{u}^2\end{aligned} \tag{48}$$

$$- a_1 a_2 \widehat{u}_2 \widehat{u}^2 + 4a_2 \widehat{u}_3 \widehat{u}^3 - a_1 a_2 \widehat{u}_3 \widehat{u}^3)(\xi_0 \xi^0 + \xi_1 \xi^1 + \xi_2 \xi^2 + \xi_3 \xi^3)^2.$$

It is not difficult to see, after some manipulations, that $\widetilde{p}_2(\widehat{U}, \xi)$ is precisely $p_2(\widehat{U}, \xi)$, i.e., (13). Let us now analyze $\widetilde{p}_3(\widehat{U}, \xi)$. The first term in $\widetilde{p}_3(\widehat{U}, \xi)$, that spans lines 2 to 5 in (C), is proportional to $(\widehat{u}^\mu \xi_\mu)^4$. The terms from lines 6 to 9 combined are also proportional to $(\widehat{u}^\mu \xi_\mu)^4$. Indeed, the term on the sixth line can be written as

$$\begin{aligned} &-2(-a_2 \widehat{u}_0 + 4a_1 a_2 \widehat{u}_0 + 3a_2^2 \widehat{u}_0)(\xi_0 \widehat{u}^0 + \xi_1 \widehat{u}^1 + \xi_2 \widehat{u}^2 + \xi_3 \widehat{u}^3)^3 \xi^0 \\ &= -2(-a_2 + 4a_1 a_2 + 3a_2^2)(\xi_0 \widehat{u}^0 + \xi_1 \widehat{u}^1 + \xi_2 \widehat{u}^2 + \xi_3 \widehat{u}^3)^3 \widehat{u}_0 \xi^0, \end{aligned}$$

and similarly we can group $\widehat{u}_i$ with ξ^i in the terms on the seventh to ninth line. Factoring then the common factor in lines 6 to 9 gives a term cubic in $\widehat{u}^\mu \xi_\mu$ times the term

$$\widehat{u}_0 \xi^0 + \widehat{u}_1 \xi^1 + \widehat{u}_2 \xi^2 + \widehat{u}_3 \xi^3.$$

But this last term equals $\widehat{u}^\mu \xi_\mu$, which can then be grouped with the cubic term in $\widehat{u}^\mu \xi_\mu$ producing a term proportional to $(\widehat{u}^\mu \xi_\mu)^4$, as claimed.

The next term in $\widetilde{p}_3(\widehat{U}, \xi)$, spanning lines 10 to 13 in (C) is proportional to $(\widehat{u}^\mu \xi_\mu)^2$.

We claim that the terms spanning lines 14 to 20, when combined, produce a term proportional to $(\widehat{u}^\mu \xi_\mu)^2$. To see this, note that as written the terms in lines 14 to 20 all have a factor $\widehat{u}_0 \xi^0 + \widehat{u}_1 \xi^1 + \widehat{u}_2 \xi^2 + \widehat{u}_3 \xi^3$, which equals $\widehat{u}^\mu \xi_\mu$. The term that begins on line 14 of (C) can be written as

$$\begin{aligned} &(3a_1 \widehat{u}_0 + 2a_2 \widehat{u}_0 + a_1 a_2 \widehat{u}_0)(\xi_0 \widehat{u}^0 + \xi_1 \widehat{u}^1 + \xi_2 \widehat{u}^2 + \xi_3 \widehat{u}^3)\xi^0(\xi_0 \xi^0 + \xi_1 \xi^1 + \xi_2 \xi^2 + \xi_3 \xi^3) \\ &= (3a_1 + 2a_2 + a_1 a_2)(\xi_0 \widehat{u}^0 + \xi_1 \widehat{u}^1 + \xi_2 \widehat{u}^2 + \xi_3 \widehat{u}^3)(\xi_0 \xi^0 + \xi_1 \xi^1 + \xi_2 \xi^2 + \xi_3 \xi^3)\widehat{u}_0 \xi^0, \end{aligned}$$

and similarly we can combine $\widehat{u}_i$ with ξ^i in the other terms in lines 15 to 20. Factoring then the common factor to all terms in lines 14 to 20 produces a term linear in $\widehat{u}^\mu \xi_\mu$ times $\widehat{u}_0 \xi^0 + \widehat{u}_1 \xi^1 + \widehat{u}_2 \xi^2 + \widehat{u}_3 \xi^3 \equiv u^\mu \xi_\mu$, hence a term quadratic in $\widehat{u}^\mu \xi_\mu$, as claimed.

Therefore, we see that all terms in $\widetilde{p}_3(\widehat{U}, \xi)$ contain a factor of $(\widehat{u}^\mu \xi_\mu)^2$, except for the last term which spans lines 21 and 22. This last term, however, vanishes identically if $a_1 = 4$. In this case we can factor $(\widehat{u}^\mu \xi_\mu)^2$ from $\widetilde{p}_3(\widehat{U}, \xi)$. We combine the factored $(\widehat{u}^\mu \xi_\mu)^2$ with $\widetilde{p}_1(\widehat{U}, \xi)$, producing $p_1(\widehat{U}, \xi)$, i.e., (12), and the remainder from $\widetilde{p}_3(\widehat{U}, \xi)$ produces $p_3(\widehat{U}, \xi)$, i.e., (14).

Remark 16. Without setting $a_1 = 4$, the above factorization procedure can be used to show that $\widetilde{p}_3(\widehat{U}, \xi)$ factors as

$$A(\widehat{u}^\mu \xi_\mu)^4 + B(\widehat{u}^\mu \xi_\mu)^2 \xi^\lambda \xi_\lambda + C(\xi^\lambda \xi_\lambda)^2,$$

where A, B, and C depend on a_1 and a_2. We would like to factor this quartic polynomial as a product of (real) degree two polynomials, since then we can analyze its roots explicitly. The above choice $a_1 = 4$ does exactly this. But other choices of a_1 and a_2 also lead to the desired factorization, as showed in [3].

Acknowledgments. The author would like to thank the anonymous referees for reading the manuscript carefully and suggesting several improvements.

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Received August 2017; revised November 2018.

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